# Multiplets of representations, twisted Dirac operators and Vogan's conjecture in affine setting

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**Abstract**

We extend classical results of Kostant et al. on multiplets of representations of finite dimensional Lie algebras and on the cubic Dirac operator to the setting of affine Lie algebras and twisted affine cubic Dirac operator. We prove in this setting an analogue of Vogan's conjecture on infinitesimal characters of Harish–Chandra modules in terms of Dirac cohomology. For our calculations we use the machinery of Lie conformal and vertex algebras.

## Contents

# 1 Introduction

Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra over $\mathbb{C}$ with a non degenerate symmetric invariant bilinear form $(\ ,\ )$, and let $\mathfrak{a}$ be a reductive subalgebra of $\mathfrak{g}$, such that the restriction of the form $(\ ,\ )$ to $\mathfrak{a}$ is non-degenerate. For future reference, we call $(\mathfrak{g},\mathfrak{a})$ a reductive pair. Then the adjoint representation of $\mathfrak{a}$ in $\mathfrak{g}$ induces a homomorphism $\mathfrak{a}\to so(\mathfrak{p})$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$. Hence, by restriction, we obtain a representation of the Lie algebra $\mathfrak{a}$ in the spinor representation $F$ of $so(\mathfrak{p})$. Assume now that $\mathrm{rank}(\mathfrak{a}) = \mathrm{rank}(\mathfrak{g})$. Then $\dim(\mathfrak{p})$ is even, hence $F$ decomposes in a direct sum of two irreducible representations $F^+$ and $F^-$ of $so(\mathfrak{p})$. The following "homogeneous Weyl character formula" was established in [3], as an identity in the character ring of $\mathfrak{a}$:

$$V(\lambda)\otimes F^+ - V(\lambda)\otimes F^- = \sum_{w\in W'}(-1)^{\ell(w)}U(w(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{a}}), \tag{1.1}$$

where $\rho_{\mathfrak{g}}, \rho_{\mathfrak{a}}$ are half the sum of the positive roots of $\mathfrak{g}$ and $\mathfrak{a}$ respectively, $V(\lambda)$ is an irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$, $U(\mu)$ is an irreducible representation of $\mathfrak{a}$ with highest weight $\mu$, and $W'$ is the set of minimal length representatives in the Weyl group of $\mathfrak{g}$ from each right coset of the Weyl group of $\mathfrak{a}$. In Remark 5.1 we adjust definitions as to make (1.1) meaningful even when the representatives are not minimal.

The collection of $\mathfrak{a}$-modules in the right-hand side of (1.1) is called a multiplet. It is clear from (1.1) that the Casimir operator of $\mathfrak{a}$ has equal eigenvalues on all modules from this multiplet, and also that the signed sum of dimensions of these modules is 0. In the special case of $\mathfrak{a} = so_9$, important for $M$-theory, these kind of multiplets have been earlier discovered by physicists, and formula (1.1) for the embedding $so_9 \subset F_4$ reproduced their observations.

In fact we observe (see Proposition 5.9) that any multiplet satisfies a stronger property, namely the signed sum of *quantum dimensions* of modules of a multiplet is zero. Recall that the quantum dimension of an $\mathfrak{a}$-module $L(\Lambda)$, where $\mathfrak{a}$ is a semisimple subalgebra[1], equals

$$\dim_q L(\Lambda) = \prod_{\alpha \in (\Delta_{\mathfrak{a}}^+)^\vee} \frac{[(\Lambda + \rho_{\mathfrak{a}}, \alpha)]_q}{[(\rho_{\mathfrak{a}}, \alpha)]_q}, \tag{1.2}$$

where $(\Delta_{\mathfrak{a}}^+)^\vee$ is the set of positive coroots of $\mathfrak{a}$ and $[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ (note that $\lim_{q \to 1} \dim_q L(\Lambda) = \dim L(\Lambda)$).

In a subsequent paper [12], Kostant showed that for a "cubic" Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{a}}$, with a cubic term associated to the fundamental 3-form of $\mathfrak{g}$, the kernel on $V(\lambda) \otimes F$ decomposes precisely in a direct sum of the $\mathfrak{a}$-modules of the multiplet:

$$Ker_{V(\lambda)\otimes F}\, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{a}} = \bigoplus_{w \in W'} U(w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{a}}). \tag{1.3}$$

Both decompositions (1.1) and (1.3) were extended to the setting of affine Lie algebras by Landweber [15]. The affine analogue of the "cubic" Dirac operator, used in [15], was introduced much earlier by Kac and Todorov in their work [10] on unitary representations of Neveu-Schwarz and Ramond superalgebras.

Another interesting application of the Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{a}}$ was a conjecture of Vogan expressing the infinitesimal character of an irreducible $(\mathfrak{g}, K)$-module $X$ of a real reductive group $G$ in terms of an $\mathfrak{a}$-type of $X$ appearing

[1] the definition of quantum dimension for an arbitrary reductive subalgebra $\mathfrak{a}$ is given in (5.17).

in the cohomology defined by $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{a}}$ (the "Dirac cohomology"), where $\mathfrak{a}$ is the complexification of the Lie algebra of $K$ (cf. Theorem V in the Introduction). This conjecture has been proved in [5] as a consequence of an algebraic statement (cf. Theorem HP below) relative to a "quadratic" Dirac operator associated to a symmetric pair $(\mathfrak{g}, \mathfrak{a})$ (i.e., $\mathfrak{a}$ is the fixed point set of an involution of $\mathfrak{g}$). Subsequently Kostant [13] observed that using his "cubic" Dirac operator yields this algebraic statement for any reductive pair.

In the present paper we define a general $\sigma$-twisted "cubic" Dirac operator in the affine setting, where $\sigma$ is a semisimple automorphism (not necessarily of finite order) of the reductive Lie algebra $\mathfrak{g}$, leaving invariant the subalgebra $\mathfrak{a}$, and extend all the results mentioned above to this setting. As in [10] we work with the "superization" $\mathfrak{g}[\xi]$ of $\mathfrak{g}$, where $\xi$ is an odd indeterminate with $\xi^2 = 0$, in order to treat the Kac-Moody and Clifford affinizations simultaneously. Also, we use the very convenient for calculations machinery of Lie conformal and vertex algebras, and their twisted representations.

In more detail, let

$$\widehat{\mathfrak{g}}^{super} = \mathfrak{g}[\xi][t, t^{-1}] \oplus \mathbb{C}\mathcal{K} \oplus \mathbb{C}\overline{\mathcal{K}}$$

be the "superization" of the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\mathcal{K}$, where $\mathcal{K}, \overline{\mathcal{K}}$ are even central elements, and the remaining commutation relations are $(a, b \in \mathfrak{g})$:

$$\begin{aligned}
[at^m, bt^n] &= [a, b]t^{m+n} + m\delta_{m,-n}(a, b)\mathcal{K}, \\
[at^m, b\xi t^n] &= [a, b]\xi t^{m+n}, \\
[a\xi t^m, b\xi t^n] &= \delta_{m,-n-1}(a, b)\overline{\mathcal{K}}.
\end{aligned}$$

Denote by $V^{k+g,1}$ the $\widehat{\mathfrak{g}}^{super}$-module, induced from the 1-dimensional representation of the subalgebra $\mathfrak{g}[\xi][t] \oplus \mathbb{C}\mathcal{K} \oplus \mathbb{C}\overline{\mathcal{K}}$ given by

$$\mathfrak{g}[\xi][t] \mapsto 0, \qquad \mathcal{K} \mapsto k + g, \qquad \overline{\mathcal{K}} \mapsto 1.$$

We assume here that the Casimir operator $Cas$ of $\mathfrak{g}$, acting on $\mathfrak{g}$, has a unique eigenvalue $2g$. (If $Cas$ has several eigenvalues, we take the tensor product of the corresponding modules, and below we take the sum of the corresponding "cubic" Dirac operators.) The module $V^{k+g,1}$ has a canonical structure of a vertex algebra with vacuum vector $|0\rangle = 1$, translation operator $T$, induced by $-\frac{d}{dt}$, and generating fields

$$\{a(z) = \sum_{j \in \mathbb{Z}} (at^j) z^{-j-1}, \ \overline{a}(z) = \sum_{j \in \mathbb{Z}} (a\xi t^j) z^{-j-1}\}_{a \in \mathfrak{g}}.$$

Denote by $a$ and $\overline{a}$ the corresponding elements under the field-state correspondence: $a = (at^{-1})|0\rangle$, $\overline{a} = (a\xi t^{-1})|0\rangle$. Let $\{x_i\}$ and $\{x^i\}$ be dual bases of $\mathfrak{g}$, i.e. $(x_i, x^j) = \delta_{i,j}$. The following element of the vertex algebra $V^{k+g,1}$ is the father of all affine $\sigma$-twisted "cubic" Dirac operators (cf. [10], [1]):

$$G_{\mathfrak{g}} = \sum_i : x_i \overline{x}^i : + \tfrac{1}{3} \sum_{i,j} : \overline{[x_i, x_j]} \overline{x}^i \overline{x}^j : .$$

This element satisfies the following key $\lambda$-bracket relation (cf. [10], [1]):

$$[G_{\mathfrak{g}\lambda} G_{\mathfrak{g}}] = L_{\mathfrak{g}} + \frac{\lambda^2}{2}(k + \frac{g}{3}) \dim \mathfrak{g}, \tag{1.4}$$

where

$$L_{\mathfrak{g}} = \sum_i : x_i x^i : + (k+g) \sum_i : T(\overline{x}_i) \overline{x}^i : + \sum_{i,j} : \overline{x}^i [x_i, x_j] \overline{x}^j : . \tag{1.5}$$

The basic object of our considerations is the element $G_{\mathfrak{g}} - G_{\mathfrak{a}}$, which is the father of all affine relative $\sigma$-twisted "cubic" Dirac operators, in the following sense. First, notice that the vertex algebra $V^{k+g,1}$ is isomorphic to the vertex algebra $V^k(\mathfrak{g}) \otimes F(\mathfrak{g})$, the isomorphism being induced by the map $x \mapsto x - \frac{1}{2} \sum_i : \overline{[x, x_i]} \overline{x}^i :$, $\overline{x} \mapsto \overline{x}$ ($x \in \mathfrak{g}$), where $V^k(\mathfrak{g})$ is the affine vertex algebra of level $k$, isomorphic to the subalgebra of $V^{k,1}$ generated by the fields $\{a(z)\}_{a \in \mathfrak{g}}$, and $F(\mathfrak{g})$ is the fermionic vertex algebra, isomorphic to the subalgebra of $V^{k,1}$ generated by the fields $\{\overline{a}(z)\}_{a \in \mathfrak{g}}$. Since the vertex algebra $V^k(\mathfrak{g})$ (resp. $F(\mathfrak{g})$) is the affine analogue of the universal enveloping algebra $U(\mathfrak{g})$ (resp. of the Clifford algebra generated by $\overline{\mathfrak{g}} = \mathfrak{g}\xi$), any (twisted) representation of the vertex algebra $V^{k+g,1}$ gives rise to a representation of the affine Lie algebra $\widehat{\mathfrak{g}}$ of the form $M \otimes F(\overline{\mathfrak{g}})$, where $M$ is a (twisted) $\widehat{\mathfrak{g}}$-module of level $k$ and $F(\overline{\mathfrak{g}})$ is the restriction of the (twisted) spinor representation of $\widehat{so(\mathfrak{g})}$ to $\widehat{\mathfrak{g}}$. (Here by twisted $\widehat{\mathfrak{g}}$-module we mean a representation of the twisted affine Lie algebra).

Denote by $\tau$ the automorphism of the Lie superalgebra $\mathfrak{g}[\xi]$, defined by being $\sigma$ on $\mathfrak{g}$ and $-\sigma$ on $\overline{\mathfrak{g}} = \mathfrak{g}\xi$. This automorphism induces an automorphism of the vertex algebra $V^{k+g,1}$, also denoted by $\tau$, such that $\tau(G_{\mathfrak{g}}) = -G_{\mathfrak{g}}$. It follows that in any twisted representation $M \otimes F(\overline{\mathfrak{g}})$ of $V^{k+g,1}$, the odd field corresponding to the element $G_{\mathfrak{g}}$ is of the form

$$Y^{M \otimes F(\overline{\mathfrak{g}})}(G_{\mathfrak{g}}, z) = \sum_{n \in \mathbb{Z}} G^{(\sigma)}_{\mathfrak{g},n} z^{-n-\frac{3}{2}}.$$

The operator $G^{(\sigma)}_{\mathfrak{g},0}$ (resp. $G^{(\sigma)}_{\mathfrak{g},0} - G^{(\sigma)}_{\mathfrak{a},0}$) is called the $\sigma$-twisted (resp. relative $\sigma$-twisted) affine Dirac operator.

One shows that the operators $G^{(\sigma)}_{\mathfrak{g},n}$, ($n \in \mathbb{Z}$) and their brackets define a representation of the Ramond superalgebra in $M \otimes F(\overline{\mathfrak{g}})$ (an easy way to do it is to use the $\lambda$-bracket relation (1.4) and similar $\lambda$-bracket relations for $[L_{\mathfrak{g}}\ {}_\lambda G_{\mathfrak{g}}]$ and $[L_{\mathfrak{g}}\ {}_\lambda L_{\mathfrak{g}}]$). In particular, one gets the formula for $(G^{(\sigma)}_{\mathfrak{g},0})^2 = \frac{1}{2}[G^{(\sigma)}_{\mathfrak{g},0}, G^{(\sigma)}_{\mathfrak{g},0}]$, similar to Kostant's formula in the finite dimensional setup [12] (going to the affine setup one should replace $Cas$ in Kostant's formula by the Virasoro operator $L^{(\sigma)}_{\mathfrak{g},0}$).

Moreover, the relative operators $G^{(\sigma)}_{\mathfrak{g},\mathfrak{a},n} = G^{(\sigma)}_{\mathfrak{g},n} - G^{(\sigma)}_{\mathfrak{a},n}$ ($n \in \mathbb{Z}$) and their brackets again define a representation of the Ramond superalgebra in $M \otimes F(\overline{\mathfrak{g}})$, which intertwines the representation of $\widehat{\mathfrak{a}}$. This construction was used in [10] to describe all unitary discrete series representations of the Ramond (and Neveu-Schwarz) superalgebra. Here we use this construction to obtain a twisted representation of $\widehat{\mathfrak{a}}$, and to derive a formula, similar to (1.3) in the affine setting. This generalizes the main result of [15] from $\sigma = 1$ to arbitrary $\sigma$.

The proof of the decomposition (1.1) in [3] works also in the affine setting (cf. [15] for $\sigma = 1$). As a corollary, we obtain that the signed sum of asymptotic dimensions of $\widehat{\mathfrak{a}}$-modules from a multiplet equals zero. Of course, all the $\widehat{\mathfrak{a}}$-modules are infinite dimensional, but one can use a substitute for dimension, called *asymptotic dimension* [8] (cf. (5.15)), which is a positive real number that has all the basic properties of dimension. Moreover, if $\mathfrak{a}$ is reductive, the sum runs over an infinite set, but a suitable use of Theta functions makes the sum finite.

Next we introduce a notion of Dirac cohomology $H((G_{\mathfrak{g},\mathfrak{a}})_0, M)$ of a $\sigma$-twisted $\widehat{\mathfrak{g}}$-module $M$. It turns out that it is not difficult to prove a non-vanishing result for this cohomology which is an affine analogue of Kostant's result [13]. In light of this, it is natural to speculate in our affine setting on results in the spirit of the following conjecture of Vogan [16], proved by Huang and Pandžić [5].

Let $G$ be a connected real reductive group with a maximal compact subgroup $K$ and let $\tilde{K}$ be the two-fold spin cover of $K$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ of $G$.

**Theorem V.** *Let $X$ be an irreducible $(\mathfrak{g}, K)$-module. Let $\gamma$ be the highest weight of an irreducible $\tilde{K}$-module appearing in the Dirac cohomology of $X$. Then the infinitesimal character of $X$ is given by $\gamma + \rho_{\mathfrak{k}}$, where $\rho_{\mathfrak{k}}$ is the half sum of the positive roots of $\mathfrak{k}$.*

For more details on this statement (in particular for an explanation of how $\gamma + \rho_{\mathfrak{k}}$ defines an infinitesimal character) see the discussion in [5, 2.3].

This theorem has been proved in [5] as a consequence of a purely algebraic statement, which is as follows. Consider the embeddings $\mathfrak{k} \to \mathfrak{g} \to U(\mathfrak{g})$, $\mathfrak{k} \to so(\mathfrak{p}) \to Cl(\mathfrak{p})$, and let $\mathfrak{k}_\Delta$ be the associated diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes Cl(\mathfrak{p})$. Let $Z(\mathfrak{g})$, $Z(\mathfrak{k}_\Delta)$ be the centers of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$, let $\mathfrak{h}$, $\mathfrak{t}$ be Cartan subalgebras of $\mathfrak{g}, \mathfrak{k}$ and $W$, $W_\mathfrak{k}$ their respective Weyl groups. Recall the isomorphisms $S(\mathfrak{h})^W \cong P(\mathfrak{h}^*)^W$, $S(\mathfrak{t})^{W_\mathfrak{k}} \cong P(\mathfrak{t}^*)^{W_\mathfrak{k}}$ (here $P(V)$ denotes the algebra of polynomial functions on the vector space $V$ and $S(V)$ the symmetric algebra on $V$).

**Theorem HP.** *Let $\{z_i\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$ and let $D = \sum_i z_i \otimes z_i$ be the Dirac operator.*

1. *For any $z \in Z(\mathfrak{g})$ there is a unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$ and an element $a \in U(\mathfrak{g}) \otimes Cl(\mathfrak{p})$ such that $z \otimes 1 = \zeta(z) + aD + Da$.*

2. *The map $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta) \cong Z(\mathfrak{k})$ is an algebra homomorphism which makes the following diagram commutative:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{k}) \\ \downarrow & & \downarrow \\ S(\mathfrak{h})^W & \xrightarrow{Res} & S(\mathfrak{t})^{W_K} \end{array}$$

*Here the vertical arrows are Harish-Chandra isomorphisms, whereas Res is the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$.*

Kostant observed in [13] that Theorem HP holds for any reductive pair $(\mathfrak{g}, \mathfrak{a})$ provided one uses the cubic Dirac operator. Since this operator is in a precise sense a specialization of our $G^{(\sigma)}_{\mathfrak{g},\mathfrak{a},0}$ (see the discussion in [1, §3.4]), it is natural to ask whether it is possible to formulate a kind of affine analogue of Theorems V or HP. We obtain this analogue in the following setting.

We replace the $(\mathfrak{g}, K)$-module $X$ by a highest weight twisted $\widehat{\mathfrak{g}}$-module $M$. Let $\mathfrak{g}^{\bar{0}}$, $\mathfrak{a}^{\bar{0}}$ denote the fixed point subalgebras of $\sigma$ in $\mathfrak{g}$, $\mathfrak{a}$ respectively. Let $\mathfrak{h}_0$ denote a Cartan subalgebra of $\mathfrak{g}^{\bar{0}}$. Fix a Cartan subalgebra $\mathfrak{h}_\mathfrak{a}$ of $\mathfrak{a}^{\bar{0}}$ and let $\widehat{\mathfrak{h}}_\mathfrak{a}$ be the corresponding Cartan subalgebra of $\widehat{\mathfrak{a}}$. Let $C_\mathfrak{g}$ denote the Tits cone of $\widehat{\mathfrak{g}}$ (cf. (8.1)) and let $\widehat{\rho}_\sigma$, $\widehat{\rho}_{\mathfrak{a}\sigma}$ be as in (4.27). Let $\widehat{W}$ be the Weyl group of $\widehat{\mathfrak{g}}$. The following result is our affine analog of Vogan's conjecture.

**Theorem 1.1.** *Assume that the centralizer $C(\mathfrak{h}_\mathfrak{a})$ of $\mathfrak{h}_\mathfrak{a}$ in $\mathfrak{g}^{\bar{0}}$ equals $\mathfrak{h}_0$. Fix $\Lambda \in \widehat{\mathfrak{h}}_0^*$ such that $\Lambda + \widehat{\rho}_\sigma \in C_\mathfrak{g}$ and let $M$ be a highest weight module for $\widehat{\mathfrak{g}}$ with highest weight $\Lambda$. Let $f$ be a holomorphic $\widehat{W}$-invariant function on $C_\mathfrak{g}$.*

*Suppose that a twisted highest weight $\widehat{\mathfrak{a}}$-module of highest weight $\mu$ occurs in the Dirac cohomology $H((G_{\mathfrak{g},\mathfrak{a}})_0, M)$. Then*

$$f_{|\widehat{\mathfrak{h}}_{\mathfrak{a}}^*}(\mu + \widehat{\rho}_{\mathfrak{a}\sigma}) = f(\Lambda + \widehat{\rho}_\sigma).$$

The content of the paper is as follows. In Sections 2 and 3 we introduce the basic material on vertex and Lie conformal algebras, and on their twisted representations, respectively. Here we discuss in some detail the examples of affine, fermionic and super affine vertex algebras, and their twisted representations.

In Section 4 we introduce (for any reductive pair) the relative affine Dirac operators, in the framework of twisted representations of super affine vertex algebras. We compute their squares and the values of the squares on highest weight vectors (formula (4.10) and Propositions 4.2 and 4.6). Also, we obtain formula (4.21), as a corollary of these computations. It is shown in Section 6 that in the special case when $\sigma$ is a finite order automorphism, formula (4.21) turns into the "very strange formula" [8, (13.15.4)].

In Section 5 we decompose (under the assumption rank $\mathfrak{a}^{\overline{0}} =$ rank $\mathfrak{g}^{\overline{0}}$) the kernel of a relative affine Dirac operator in the multiplets (Theorem 5.4), compute the (common) eigenvalue of the affine Casimir operator on representations of each multiplet (Corollary 5.6) and show that the signed sum of asymptotic dimensions of representations of a multiplet is zero (Proposition 5.7).

In Section 7, in the general setting of reductive pairs, we obtain a nonvanishing result for affine Dirac cohomology, similar to Kostant's [13] in the finite dimensional setting.

In by far the longest Section 8 we prove our affine analogue of Vogan's conjecture: the main result is Theorem 8.1, which is a technically more precise formulation of Theorem 1.1 above. Though the flavor of Huang-Pandžić's proof remains (notably in exploiting the exactness of suitable Koszul complexes), we have to overcome several difficulties which are due to the completion, which has to be introduced in order to have a large holomorphic center, constructed in [7], and the corresponding Harish-Chandra type homomorphism.

In Section 9 we give proofs, omitted in previous sections, of various technical results.

## 2 Basic definitions and examples

For background on vertex algebras, conformal Lie algebras and twisted vertex algebras see [9], [1], [11].

A vector superspace is a $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If $\alpha \in \mathbb{Z}/2\mathbb{Z}$, then set $p(v) = \alpha$ if $v \in V_\alpha$ and call $p(v)$ the parity of $v$. We also set $p(v, w) = (-1)^{p(v)p(w)}$. Recall that an $End(V)$-valued quantum field is a series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

where $a_{(n)} \in End(V)$ all have the same parity (called the parity of $a(z)$) and, for all $v \in V$, $a_{(n)}v = 0$ for $n >> 0$.

**Definition 2.1.** A *vertex algebra* is triple $(V, |0\rangle, Y)$, where $V$ is a vector superspace, $|0\rangle$ is an even vector in $V$, and $Y : a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ is a parity preserving linear map from $V$ to the space of $End(V)$-valued quantum fields. These data satisfy the following axioms, where $Ta = a_{(-2)}|0\rangle$:

i) $T|0\rangle = 0$, $Y(a, z)|0\rangle_{|z=0} = a$,

ii) $[T, Y(a, z)] = \partial_z Y(a, z)$,

iii) $(z - w)^N [Y(a, z), Y(b, w)] = 0$ for some $N \in \mathbb{Z}_+$.

As a consequence of the axioms one deduces that, if $a, b \in V$, then, in $End(V)$,

$$[a_{(n)}, b_{(m)}] = \sum_{j \in \mathbb{Z}_+} \binom{n}{j} (a_{(j)}b)_{(n+m-j)}, \quad n, m \in \mathbb{Z}. \tag{2.1}$$

In a vertex algebra one defines a bilinear product $: \cdot :$, called the normal order product, by $: ab := a_{(-1)}b$. Letting $Y^+(a, z) = \sum_{n<0} a_{(n)} z^{-n-1}$ and $Y^-(a, z) = \sum_{n \geq 0} a_{(n)} z^{-n-1}$, one defines the normal order product of quantum fields by

$$: Y(a, z)Y(b, z) := Y^+(a, z)Y(b, z) + p(a, b)Y(b, z)Y^-(a, z).$$

Then

$$: Y(a, z)Y(b, z) := Y(: ab :, z).$$

The normal order product is, in general, neither commutative nor associative, but the following "quasi-commutativity" and "quasi-associativity" relations hold:

$$: ab := p(a, b) : ba : + \int_{-T}^{0} [a_\lambda b] d\lambda. \tag{2.2}$$

$$:: ab : c :=: a : bc :: + : (\int_0^T d\lambda\, a)[b_\lambda c] : + p(a, b) : (\int_0^T d\lambda\, b)[a_\lambda c] : . \tag{2.3}$$

where

$$[a_\lambda b] = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)} b. \tag{2.4}$$

**Definition 2.2.** A *Lie conformal superalgebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[T]$-module $R = R_{\bar{0}} \oplus R_{\bar{1}}$, endowed with a parity preserving $\mathbb{C}$-bilinear map $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, denoted by $[a_\lambda b]$, such that the following axioms hold:

$$\begin{array}{rl} \text{(sesquilinearity)} & [Ta_\lambda b] = -\lambda[a_\lambda b],\ T[a_\lambda b] = [Ta_\lambda b] + [a_\lambda Tb], \\ \text{(skewsymmetry)} & [b_\lambda a] = -p(a,b)[a_{-\lambda-T} b], \\ \text{(Jacobi identity)} & [a_\lambda[b_\mu c]] - p(a,b)[b_\mu[a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]. \end{array}$$

A vertex algebra can be endowed with the structure of a Lie conformal superalgebra by introducing the $\lambda$-product via (2.4). Moreover $[\cdot_\lambda \cdot]$ and $:\cdot\,\cdot:$ are related by the *non-commutative Wick formula* :

$$[a_\lambda :bc:] =: [a_\lambda b]c: + p(a,b) :b[a_\lambda c]: + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu. \tag{2.5}$$

Combining Wick formula with skewsymmetry we get the "right non-commutative Wick formula":

$$[:ab:_\lambda c] =: (e^{T\partial_\lambda} a)[b_\lambda c]: + p(a,b) : (e^{T\partial_\lambda} b)[a_\lambda c]: + p(a,b) \int_0^\lambda [b_\mu [a_{\lambda-\mu} c]] d\mu. \tag{2.6}$$

Given a Lie conformal superalgebra $R$ one can construct its universal enveloping vertex algebra $V(R)$. This vertex algebra is characterized by the following properties:

1. There is an embedding $R \to V(R)$ of Lie conformal superalgebras.

2. Given an ordered basis $\{a_i\}$ of $R$, the monomials $: a_{i_1} a_{i_2} \cdots a_{i_n} :$ with $i_j \leq i_{j+1}$ and $i_j < i_{j+1}$ if $p(a_{i_j}) = \bar{1}$ form a basis of $V(R)$.

Here and further, the normal ordered product of more than two fields is defined from right to left, as usual.

We will be using the following three examples of Lie conformal superalgebras and their universal enveloping vertex algebras.

## 2.1 The affine vertex algebra

Given a reductive finite dimensional complex Lie algebra $\mathfrak{g}$ endowed with a nondegenerate invariant bilinear form $(\cdot,\cdot)$, one defines the current Lie conformal algebra $Cur(\mathfrak{g})$ as

$$Cur(\mathfrak{g}) = (\mathbb{C}[T] \otimes \mathfrak{g}) + \mathbb{C}K$$

with $T(K) = 0$ and the $\lambda$-bracket defined for $a, b \in 1 \otimes \mathfrak{g}$ by

$$[a_\lambda b] = [a, b] + \lambda(a, b)K, \quad [a_\lambda K] = [K_\lambda K] = 0.$$

Let $V(\mathfrak{g})$ be its universal enveloping vertex algebra. Given $a, b \in \mathfrak{g}$ then it follows from (2.1) that, in $End(V(\mathfrak{g}))$,

$$[a_{(n)}, b_{(m)}] = [a, b] + \delta_{n,m} n(a, b)K, \quad m, n \in \mathbb{Z}.$$

The vertex algebra

$$V^k(\mathfrak{g}) = V(\mathfrak{g}) / :(K - k|0\rangle)V(\mathfrak{g}):$$

is called the *level $k$ universal affine vertex algebra*.

## 2.2 The fermionic vertex algebra

Given a vector superspace $A$ with a nondegenerate bilinear form $(\cdot, \cdot)$ such that $(a, b) = (-1)^{p(a)}(b, a)$, one can construct the Clifford Lie conformal algebra as

$$R^{Cl}(A) = (\mathbb{C}[T] \otimes A) \oplus \mathbb{C}K'$$

with $T(K') = 0$ and the $\lambda$-bracket defined by

$$[a_\lambda b] = (a, b)K', \quad [a_\lambda K'] = [K'_\lambda K'] = 0.$$

Let $V(A)$ be its universal enveloping vertex algebra. Given $a, b \in A$, it follows from (2.1) that, in $End(V(A))$,

$$[a_{(n)}, b_{(m)}] = \delta_{n+m,-1}(a, b)K', \quad m, n \in \mathbb{Z}.$$

The vertex algebra

$$F(A) = V(A) / :(K' - |0\rangle)V(A):$$

is called the *fermionic vertex algebra*.

## 2.3 The super affine vertex algebra

Let $\mathfrak{g}$ be a reductive complex finite dimensional Lie algebra endowed with a non-degenerate symmetric bilinear invariant form $(\cdot, \cdot)$. Regard $\mathfrak{g}$ as an even superspace and let $\overline{\mathfrak{g}}$ be $\mathfrak{g}$ viewed as an odd superspace. Consider the Lie conformal superalgebra $R^{super} = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}\mathcal{K} \oplus \mathbb{C}\overline{\mathcal{K}}$

with $T(\mathcal{K}) = T(\overline{\mathcal{K}}) = 0$, $\mathcal{K}$, $\overline{\mathcal{K}}$ being even central elements, the $\lambda$-brackets for $a, b \in 1 \otimes \mathfrak{g}$ being

$$[a_\lambda b] = [a, b] + \lambda(a, b)\mathcal{K}, \quad [a_\lambda \overline{b}] = [\overline{a}_\lambda b] = \overline{[a, b]}, \quad [\overline{a}_\lambda \overline{b}] = (a, b)\overline{\mathcal{K}}. \tag{2.7}$$

Denote by $V(R^{super})$ the corresponding universal enveloping vertex algebra, by $V^1(R^{super})$ its quotient modulo the ideal generated by $\overline{\mathcal{K}} - |0\rangle$ and by $V^{k,1}(R^{super})$ the quotient modulo the ideal generated by $\mathcal{K} - k|0\rangle$ and $\overline{\mathcal{K}} - |0\rangle$.

We conclude this section by showing how our three examples are related. First of all we recall that a tensor product of vertex algebras $V_1$ and $V_2$ is the vertex algebra $V_1 \otimes V_2$ with vacuum vector $|0\rangle_1 \otimes |0\rangle_2$ and state-field correspondence defined by $Y(a \otimes b, z) = Y(a, z) \otimes Y(b, z)$.

Assume that $\mathfrak{g}$ is semisimple or abelian. Let $Cas$ be the Casimir operator of $\mathfrak{g}$ with respect to $(\cdot, \cdot)$. We can and do assume that the form is chosen so that $Cas = 2\, g\, I_{\mathfrak{g}}$ when acting on $\mathfrak{g}$, where $g$ is a positive real number.

**Proposition 2.1.** *Let $\{x_i\}$ be an orthonormal basis of $\mathfrak{g}$. For $x \in \mathfrak{g}$ set*

$$\widetilde{x} = x - \frac{1}{2} \sum_i : \overline{[x, x_i]}\overline{x}_i :, \ \ \widetilde{K} = \mathcal{K} - g|0\rangle.$$

*The map $x \mapsto \widetilde{x}$, $K \mapsto \widetilde{K}$, $\overline{y} \mapsto \overline{y}$ defines a Lie conformal superalgebra homomorphism $Cur(\mathfrak{g}) \oplus R^{Cl}(\overline{\mathfrak{g}}) \to V^1(R^{super})$, which induces an isomorphism of vertex algebras $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}}) \cong V^{k+g,1}(R^{super})$.*

*Proof.* We first show that for $a, b \in \mathfrak{g}$ we have (in $V^1(R^{super})$)

$$[a_\lambda \widetilde{b}] = \widetilde{[a, b]} + \lambda(\mathcal{K} - g|0\rangle)(a, b) = [\widetilde{a}_\lambda b]. \tag{2.8}$$

The first equality in (2.8) follows from the Wick formula (2.5) and the Jacobi identity:

$$\begin{aligned}
[a_\lambda \widetilde{b}] &= [a_\lambda b] - \tfrac{1}{2} \sum_i [a_\lambda : \overline{[b, x_i]}\overline{x}_i :] \\
&= [a_\lambda b] - \tfrac{1}{2} \sum_i (: [a_\lambda \overline{[b, x_i]}]\overline{x}_i : + : \overline{[b, x_i]}[a_\lambda \overline{x}_i] : + \int_0^\lambda [[a_\lambda \overline{[b, x_i]}]_\mu \overline{x}_i] d\mu) \\
&= [a_\lambda b] - \tfrac{1}{2} \sum_i (: \overline{[a, [b, x_i]]}\overline{x}_i : + : \overline{[b, x_i]}\ \overline{[a, x_i]} : + \lambda([a, [b, x_i]], x_i)) \\
&= [a_\lambda b] - \tfrac{1}{2} \sum_i (: \overline{[a, [b, x_i]]}\overline{x}_i : + : \overline{[b, [x_i, a]]}\overline{x}_i :) - \lambda g(a, b)|0\rangle \\
&= [a_\lambda b] - \tfrac{1}{2} \sum_i : \overline{[[a, b], x_i]}\overline{x}_i : - \lambda g(a, b)|0\rangle.
\end{aligned}$$

Recalling that $[a_\lambda b] = [a,b] + \lambda(a,b)\mathcal{K}$ we have (2.8). By skewsymmetry of the $\lambda$-bracket, we readily obtain the second equality in (2.8).

Next we prove that

$$[\overline{a}_\lambda \widetilde{b}] = 0. \tag{2.9}$$

Just compute, using the Wick formula (2.5):

$$\begin{aligned}
[\overline{a}_\lambda \widetilde{b}] &= [\overline{a}_\lambda b] - \tfrac{1}{2}\sum_i [\overline{a}_\lambda : \overline{[b,x_i]}\overline{x}_i :] \\
&= \overline{[a,b]} - \tfrac{1}{2}(\sum_i : [\overline{a}_\lambda \overline{[b,x_i]}]\overline{x}_i : - \sum_i : \overline{[b,x_i]}[\overline{a}_\lambda \overline{x}_i] :) \\
&= \overline{[a,b]} - \tfrac{1}{2}(\sum_i (a,[b,x_i])\overline{x}_i - \sum_i (a,x_i)\overline{[b,x_i]}) \\
&= \overline{[a,b]} - \tfrac{1}{2}\overline{[a,b]} + \tfrac{1}{2}\overline{[b,a]} = 0.
\end{aligned}$$

Finally, using (2.9) we get $[\widetilde{a}_\lambda \widetilde{b}] = [a_\lambda \tilde{b}]$, and, using (2.8), we find

$$[\widetilde{a}_\lambda \widetilde{b}] = \widetilde{[a,b]} + \lambda(\mathcal{K} - g)(a,b) = \widetilde{[a,b]} + \lambda\widetilde{K}(a,b) = \widetilde{[a_\lambda b]}. \tag{2.10}$$

This proves the first part of the statement.

Clearly the ideal generated by $K - k|0\rangle$ gets mapped to the ideal of $V^1(R^{super})$ generated by $\mathcal{K} - (k+g)|0\rangle$, so our map factors to a map from $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$ to $V^{k+g,1}(R^{super})$. We now show that this map is an isomorphism. For this it suffices to show that it maps a basis of $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$ to a basis of $V^{k+g,1}(R^{super})$. A basis of $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$ is given by vectors

$$: T^{i_1}(x_{j_1}) \cdots T^{i_h}(x_{j_h}) : \otimes : T^{r_1}(\overline{x}_{s_1}) \cdots T^{r_t}(\overline{x}_{s_t}) :,$$

with $j_1 \leq \ldots \leq j_k$, $s_1 < \ldots < s_l$. These map to

$$: T^{i_1}(\tilde{x}_{j_1}) \cdots T^{i_h}(\tilde{x}_{j_h}) T^{r_1}(\overline{x}_{s_1}) \cdots T^{r_t}(\overline{x}_{s_t}) : .$$

Define a filtration of $V^{k+g,1}(R^{super})$ (as a vector space) by setting

$$V^{k+g,1}(R^{super})_m = span\{: T^{i_1}(x_{j_1}) \cdots T^{i_h}(x_{j_h}) T^{r_1}(\overline{x}_{s_1}) \cdots T^{r_t}(\overline{x}_{s_t}) :| \, h \leq m\}.$$

Then, since $[\tilde{x}_\lambda \overline{y}] = 0$, we see that

$$\begin{aligned}
&: T^{i_1}(\tilde{x}_{j_1}) \cdots T^{i_h}(\tilde{x}_{j_h}) T^{r_1}(\overline{x}_{s_1}) \cdots T^{r_t}(\overline{x}_{s_t}) : \\
&\qquad =: T^{i_1}(x_{j_1}) \cdots T^{i_h}(x_{j_h}) T^{r_1}(\overline{x}_{s_1}) \cdots T^{r_t}(\overline{x}_{s_t}) : + a,
\end{aligned}$$

with $a \in V^{k+g,1}(R^{super})_{h-1}$. Since the vectors

$$: T^{i_1}(x_{j_1}) \cdots T^{i_h}(x_{j_h}) T^{r_1}(\overline{x}_{s_1}) \cdots T^{r_t}(\overline{x}_{s_t}) :$$

form a basis of $V^{k+g,1}(R^{super})$ we are done.

$\square$

# 3 Representations of vertex algebras

A *field module over a Lie conformal superalgebra* $R$ is a vector superspace $M$ endowed with a linear map $Y^M$ from $R$ to the superspace of $End(M)$-valued quantum fields,

$$a \mapsto Y^M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$$

such that for all $a, b \in R$, $m, n \in \mathbb{Z}$ one has:

$$[a^M_{(n)}, b^M_{(m)}] = \sum_{j\in\mathbb{Z}_+} \binom{n}{j} (a_{(j)}b)^M_{(n+m-j)}, \tag{3.1}$$

$$Y^M(T(a), z) = \partial_z Y^M(a,z). \tag{3.2}$$

Also recall that a representation of a vertex algebra $V$ in a vector superspace $M$ is a linear map $Y^M : V \to End(M)[[z, z^{-1}]]$ as above, such that $Y^M(|0\rangle, z) = I_M$, $: Y^M(a,z)Y^M(b,z) := Y^M(: ab :, z)$ and (3.1), (3.2) hold. Notice that the vertex algebra $V$ itself is a representation of $V$.

More generally, one has the notion of a twisted field module over a Lie conformal superalgebra and of a twisted representation of a vertex algebra. Let $R$ be a Lie conformal algebra and $\sigma$ a semisimple automorphism of $R$. Assume (for simplicity) that the eigenvalues of $\sigma$ are of modulus one. Then $R$ decomposes in a direct sum of $\mathbb{C}[T]$-submodules $R^{\overline{\mu}} = \{a \in R \mid \sigma(a) = e^{2\pi i\overline{\mu}}a\}$, $\overline{\mu} \in \mathbb{R}/\mathbb{Z}$. A $\sigma$-twisted field module over $R$ is a vector superspace $M$ endowed with a linear map $Y^M : a \mapsto Y^M(a,z)$, where $Y^M(a,z)$ with $a \in R^{\overline{\mu}}$ is an $End(M)$-valued $\sigma$-twisted quantum field

$$Y^M(a,z) = \sum_{n\in\overline{\mu}} a^M_{(n)} z^{-n-1}, \;\; a^M_{(n)}v = 0 \;\text{ for } n >> 0,$$

such that (3.1) and (3.2) hold. Here and further $\overline{\mu}$ stands for a coset of $\mathbb{R}/\mathbb{Z}$.

Let $V$ be a vertex algebra and $\sigma$ a semisimple automorphism of $V$ with modulus one eigenvalues. If $M$ is a $\sigma$-twisted field module of $V$, viewed as a Lie conformal superalgebra, and $a \in V^{\overline{\mu}}$, choose $\mu \in \mathbb{R}$ in the coset $\overline{\mu}$ and set

$$Y^M_+(a,z) = \sum_{n<\mu} a^M_{(n)} z^{-n-1},\; Y^M_-(a,z) = \sum_{n\geq\mu} a^M_{(n)} z^{-n-1}.$$

Define the normal ordered product of $\sigma$-twisted fields as

$$: Y^M(a,z)Y^M(b,z) := Y^M_+(a,z)Y^M(b,z) + p(a,b)Y^M(b,z)Y^M_-(a,z),$$

(it depends on the choice of $\mu$ in $\overline{\mu}$).

A $\sigma$-twisted representation of a vertex algebra $V$ is a $\sigma$-twisted field module $M$ of $V$ (viewed as a Lie conformal superalgebra), such that

$$Y^M(|0\rangle, z) = I_M, \tag{3.3}$$

$$:Y^M(a,z)Y^M(b,z): = \sum_{j\in\mathbb{Z}_+} \binom{\mu}{j} Y^M(a_{(j-1)}b, z)z^{-j}, \quad a \in V^{\overline{\mu}}. \tag{3.4}$$

If $\sigma$ is an automorphism of a Lie conformal algebra $R$ then we can extend $\sigma$ to an automorphism, still denoted by $\sigma$, of $V(R)$ by

$$\sigma(:a_{i_1}\cdots a_{i_k}:) =: \sigma(a_{i_1})\cdots\sigma(a_{i_k}): .$$

The following lemma is a restatement of Proposition 1.1 of [11], which provides a handy way to construct $\sigma$-twisted representations of $V(R)$ from $\sigma$-twisted field modules over $R$.

**Lemma 3.1.** *Any $\sigma$-twisted field module over a Lie conformal algebra $R$ extends uniquely to a $\sigma$-twisted representation over $V(R)$, using* (3.3) *and* (3.4).

We now apply Lemma 3.1 to our examples of vertex algebras.

## 3.1 Twisted representations of the affine vertex algebra

Let $\sigma$ be a semisimple automorphism of $\mathfrak{g}$ with modulus 1 eigenvalues that keeps the bilinear form invariant. Then $\sigma$ can be viewed as an automorphism of $Cur(\mathfrak{g})$ by setting $\sigma(K) = K$ and letting $\sigma$ and $T$ commute. It follows that we can extend $\sigma$ to an automorphism of $V(\mathfrak{g})$ that clearly stabilizes $:(K - k|0\rangle)V(\mathfrak{g}):$. We obtain therefore an automorphism of $V^k(\mathfrak{g})$.

Let $L'(\mathfrak{g},\sigma) = \sum_{j\in\mathbb{R}}(t^j \otimes \mathfrak{g}^{\overline{j}}) \oplus \mathbb{C}K$, where $\mathfrak{g}^{\overline{j}} = \{x \in \mathfrak{g} \mid \sigma(x) = e^{2\pi i \overline{j}}x\}$, and, as before, $\overline{j} \in \mathbb{R}/\mathbb{Z}$ denotes the coset containing $j$. This is a Lie algebra with bracket defined by

$$[t^m \otimes a, t^n \otimes b] = t^{m+n} \otimes [a,b] + \delta_{m,n} m(a,b)K, \quad m,n \in \mathbb{R},$$

$K$ being a central element. We say that a $L'(\mathfrak{g},\sigma)$-module $M$ is *restricted* if, for any $v \in M$, $(t^j \otimes a)(v) = 0$ for $j >> 0$. We say that $M$ is a representation of level $k$ if $Kv = kv$ for all $v \in M$.

If $(\pi, M)$ is a restricted $L'(\mathfrak{g},\sigma)$-module of level $k$ and $a \in \mathfrak{g}^{\overline{j}}$, then define $Y^M(a,z) = \sum_{n\in\overline{j}} \pi(t^n \otimes a)z^{-n-1}$ and $Y^M(K,z) = kI_M$. Clearly these

fields satisfy (3.1) and (3.2). Applying Lemma 3.1 we obtain a $\sigma$-twisted representation of $V^k(\mathfrak{g})$ on $M$.

A particular example of a restricted module is given by a highest weight module. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}^{\overline{0}}$ and $\mathfrak{h}' = \mathfrak{h}_0 + \mathbb{C}K$. If $\mu \in (\mathfrak{h}')^*$, we set $\overline{\mu} = \mu_{|\mathfrak{h}_0}$. Denote by $\Delta_0$ the set of roots for the pair $(\mathfrak{g}^{\overline{0}}, \mathfrak{h}_0)$ and fix a subset of positive roots $\Delta_0^+$ in $\Delta_0$. For $\alpha \in \Delta_0$, let $(\mathfrak{g}^{\overline{0}})_\alpha$ denote the corresponding root space and set $\mathfrak{n} = \sum_{\alpha\in\Delta_0^+}(\mathfrak{g}^{\overline{0}})_\alpha$, $\mathfrak{n}' = \mathfrak{n} + \sum_{j>0} t^j \otimes \mathfrak{g}^{\overline{j}}$. Fix $\Lambda \in (\mathfrak{h}')^*$ and set $k = \Lambda(K)$. A $L'(\mathfrak{g},\sigma)$-module $M$ is called a highest weight module with highest weight $\Lambda$ if there is a nonzero vector $v_\Lambda \in M$ such that

$$\mathfrak{n}'(v_\Lambda) = 0, \;\; hv_\Lambda = \Lambda(h)v_\Lambda \text{ for } h \in \mathfrak{h}', \;\; U(L'(\mathfrak{g},\sigma))v_\Lambda = M. \tag{3.5}$$

If $\mu \in \mathfrak{h}_0^*$, we let $h_\mu$ be the unique element of $\mathfrak{h}_0$ such that $(h, h_\mu) = \mu(h)$. Let $\Delta_{\overline{j}}$ be the set of $\mathfrak{h}_0$-weights of $\mathfrak{g}^{\overline{j}}$. Set

$$\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha, \quad \rho_{\overline{j}} = \frac{1}{2}\sum_{\alpha\in\Delta_{\overline{j}}}(\dim \mathfrak{g}^{\overline{j}}_\alpha)\alpha \quad \text{if } \overline{j}\neq\overline{0}, \quad \rho_\sigma = \sum_{0\le j\le\frac{1}{2}}(1-2j)\rho_{\overline{j}}. \tag{3.6}$$

Choose an orthonormal basis $\{x_i\}$ of $\mathfrak{g}$ and set

$$L^{\mathfrak{g}} = \frac{1}{2}\sum_i : x_i x_i :\in V^k(\mathfrak{g}), \tag{3.7}$$

$$z(\mathfrak{g},\sigma) = \frac{1}{2}\sum_{0\le j<1}\frac{j(1-j)}{2}\dim \mathfrak{g}^{\overline{j}}. \tag{3.8}$$

**Lemma 3.2.** *If $M$ is a highest weight module over $L'(\mathfrak{g},\sigma)$ with highest weight $\Lambda$, then*

$$(L^{\mathfrak{g}})^M_{(1)}(v_\Lambda) = \frac{1}{2}(\overline{\Lambda} + 2\rho_\sigma, \overline{\Lambda})v_\Lambda + kz(\mathfrak{g},\sigma)v_\Lambda. \tag{3.9}$$

*Proof.* If $\{y_i\}$ is any basis of $\mathfrak{g}$ and $\{y^i\}$ is its dual basis, then, clearly, $2L^{\mathfrak{g}} = \sum_i : y_i y^i :$. We can and do choose $\{y_i\}$ so that $y_i \in \mathfrak{g}^{\overline{s}_i}$, for some $\overline{s}_i \in \mathbb{R}/\mathbb{Z}$. By (3.4) we have

$$\begin{aligned}\left(\sum_i (: y_i y^i :)\right)^M_{(1)} &= \sum_i\left(\sum_{n<s_i}(y_i)^M_{(n)}(y^i)^M_{(-n)} + \sum_{n\ge s_i}(y^i)^M_{(-n)}(y_i)^M_{(n)}\right)\\ &\quad - \sum_{r\in\mathbb{Z}_+}\binom{s_i}{r+1}((y_i)_{(r)}(y^i))^M_{(-r)}.\end{aligned} \tag{3.10}$$

We choose $s_i \in [0,1)$, thus

$$\left(\sum_i (: y_i y^i :)\right)^M_{(1)} (v_\Lambda) = \sum_i ((y^i)^M_{(-s_i)}(y_i)^M_{(s_i)} - s_i[y_i, y^i]^M_{(0)} - k\binom{s_i}{2})(v_\Lambda)$$

as in [11, (1.15)]. Write

$$\begin{aligned}\sum_i ((y^i)^M_{(-s_i)}(y_i)^M_{(s_i)} - s_i[y_i, y^i]^M_{(0)} - k\binom{s_i}{2})(v_\Lambda) &= \sum_{i:s_i=0} (y^i)^M_{(0)}(y_i)^M_{(0)}(v_\Lambda) \\ &\quad - \sum_{i:s_i>0} (s_i[y_i, y^i]^M_{(0)} + k\binom{s_i}{2})(v_\Lambda).\end{aligned}$$

Choosing an orthonormal basis $\{h_i\}$ of $\mathfrak{h}_0$ and writing $\sum\limits_{i:s_i=0} y^i y_i = 2h_{\rho_0} + \sum_i h_i^2 + 2\sum_{\alpha\in\Delta_0^+} X_{-\alpha}X_\alpha$, we find that

$$\begin{aligned}\left(\sum_i (: y_i y^i :)^M\right)_{(1)} (v_\Lambda) &= (\overline{\Lambda} + 2\rho_0, \overline{\Lambda})v_\Lambda + k(\sum_{0<j<1} \frac{j(1-j)}{2} \dim \mathfrak{g}^j)v_\Lambda \\ &\quad - \sum_{i:s_i>0} s_i[y_i, y^i]^M_{(0)}(v_\Lambda).\end{aligned}$$

In order to evaluate $\sum_{i:s_i>0} s_i[y_i, y^i]^M_{(0)}(v_\Lambda)$, we observe that

$$\sum_{i:s_i=s} [y_i, y^i] = \sum_{i:s_i=1-s} [y^i, y_i].$$

This relation is easily derived by exchanging the roles of $y_i$ and $y^i$. Hence

$$\begin{aligned}\sum_{i:s_i>0} s_i[y_i, y^i]^M_{(0)}(v_\Lambda) &= \sum_{i:\frac12>s_i>0} s_i[y_i, y^i]^M_{(0)}(v_\Lambda) + \sum_{i:1>s_i>\frac12} s_i[y_i, y^i]^M_{(0)}(v_\Lambda) \\ &= \sum_{i:\frac12>s_i>0} s_i[y_i, y^i]^M_{(0)}(v_\Lambda) + \sum_{i:0<s_i<\frac12} (s_i-1)[y_i, y^i]^M_{(0)}(v_\Lambda) \\ &= -\sum_{i:\frac12>s_i>0} (1-2s_i)[y_i, y^i]^M_{(0)}(v_\Lambda).\end{aligned}$$

We can choose $y_i \in \mathfrak{g}_\alpha^{\overline{s}_i}$ so that $[y_i, y^i] = h_\alpha$, hence

$$\sum_{i:\frac12>s_i>0} (1-2s_i)[y_i, y^i]^M_{(0)}(v_\Lambda) = \sum_{i:0<s_i<\frac12} (1-2s_i)(2\rho_{s_i}, \overline{\Lambda})v_\Lambda.$$

This completes the proof of (3.9). $\square$

We extend the Lie algebra $L'(\mathfrak{g},\sigma)$ by setting $\widehat{L}(\mathfrak{g},\sigma) = L'(\mathfrak{g},\sigma)\oplus\mathbb{C}d$, where $d$ is the derivation of $L'(\mathfrak{g},\sigma)$ such that $d(K)=0$ and $d$ acts as $t\frac{d}{dt}$ on $L(\mathfrak{g},\sigma)$. Set $\widehat{\mathfrak{h}}_0 = \mathfrak{h}_0\oplus\mathbb{C}K\oplus\mathbb{C}d$. If $\Lambda\in\widehat{\mathfrak{h}}_0^*$, a $\widehat{L}(\mathfrak{g},\sigma)$-module $M$ is called a highest weight module with highest weight $\Lambda$ if $M$ is a highest weight module for $L'(\mathfrak{g},\sigma)$ with highest weight $\Lambda_{|\mathfrak{h}'}$ and $d\cdot v_{\Lambda_{|\mathfrak{h}'}} = \Lambda(d)v_{\Lambda_{|\mathfrak{h}'}}$. We let $d^M$ be the operator on $M$ given by the action of $d$.

**Lemma 3.3.** *If $M$ is a highest weight module over $\widehat{L}(\mathfrak{g},\sigma)$ with highest weight $\Lambda$ and level $k$, then, as an operator on $M$,*

$$(L^{\mathfrak{g}})^M_{(1)}+(k+g)d^M = \left(\frac{1}{2}(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda})+kz(\mathfrak{g},\sigma)+(k+g)\Lambda(d)\right)I_M. \quad (3.11)$$

*Proof.* It is well known (and easy to show) that if $x\in\mathfrak{g}$, then $[x_\lambda L^{\mathfrak{g}}] = (k+g)\lambda x$, hence, by (3.1)

$$[x^M_{(n)},(L^{\mathfrak{g}})^M_{(1)}] = (k+g)nx^M_{(n)}. \quad (3.12)$$

It follows that, as operators on $M$,

$$[t^n\otimes x,(L^{\mathfrak{g}})^M_{(1)}+(k+g)d^M] = 0. \quad (3.13)$$

By Lemma 3.2,

$$((L^{\mathfrak{g}})^M_{(1)}+(k+g)d^M)\cdot v_\Lambda = (\frac{1}{2}(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda})+kz(\mathfrak{g},\sigma)+(k+g)\Lambda(d))v_\Lambda.$$

Since $M=U(L'(\mathfrak{g},\sigma))\cdot v_\Lambda$, (3.13) implies the result. □

## 3.2 Twisted representations of the fermionic vertex algebra

Analogously to the affine vertex algebra case, if $A$ is an odd vector superspace with a non-degenerate bilinear symmetric form $(\cdot,\cdot)$ and $\sigma$ is a semisimple automorphism of $A$ with modulus one eigenvalues that keeps the bilinear form invariant, then we can extend $\sigma$ to $R^{Cl}(A)$ by letting $T$ and $\sigma$ commute and setting $\sigma(K')=K'$. As in the affine case, we can extend $\sigma$ to $F(A)$.

We set $L(A,\sigma)=\oplus_{\mu\in\mathbb{R}}(t^\mu\otimes A^{\overline{\mu}})$ and define the bilinear form $<\cdot,\cdot>$ on $L(A,\sigma)$ by setting $<t^\mu\otimes a,t^\nu\otimes b> = \delta_{\mu+\nu,-1}(a,b)$. Let $Cl(L(A,\sigma))$ be the corresponding Clifford algebra. We choose a maximal isotropic subspace $L^+(A,\sigma)$ of $L(A,\sigma)$ as follows: fix a $\sigma$-invariant maximal isotropic subspace $A^+$ of $A^{-\overline{\frac{1}{2}}}$, and let

$$L^+(A,\sigma) = (\oplus_{\mu>-\frac{1}{2}}(t^\mu\otimes A^{\overline{\mu}}))\oplus(t^{-\frac{1}{2}}\otimes A^+).$$

We obtain a Clifford module $F^\sigma(A) = Cl(L(A,\sigma))/Cl(L(A,\sigma))L^+(A,\sigma)$. Then we can define fields $Y^\sigma(a,z) = \sum_{n\in\overline{\mu}}(t^n\otimes a)z^{-n-1}, a\in\ A^{\overline{\mu}}$, where we let $t^n\otimes a$ act on $F^\sigma(A)$ by left multiplication. Set $Y^\sigma(K',z) = I_{F^\sigma(A)}$. Lemma 3.1 now gives a $\sigma$-twisted representation of $F(A)$ on $F^\sigma(A)$. If $a\in F(A)$ then we write $a^\sigma_{(n)}$ instead of $a^{F^\sigma(A)}_{(n)}$. Fix a basis $\{b_i\}$ of $A$ and let $\{b^i\}$ be its dual basis. Set

$$L^A = -\frac{1}{2}\sum_i :T(b_i)b^i:\in F(A). \tag{3.14}$$

It is well known (and easy to compute) that

$$[{L^A}_\lambda a] = -(T+\frac{1}{2}\lambda)a \tag{3.15}$$

for $a\in A$, hence, by (3.1),

$$[L^A_{(1)}, a_{(n)}] = (n+\frac{1}{2})a_{(n)} \tag{3.16}$$

As in [11, (1.16)], we have from (3.4):

$$Y^\sigma(L^A,z) = -\frac{1}{2}(\sum_i :Y^\sigma(T(b_i),z)Y^\sigma(b^i,z): + \binom{s_i}{2}z^{-2}).$$

## 3.3 Twisted representations of the super affine vertex algebra

Fix, once and for all, a semisimple automorphism $\sigma$ of $\mathfrak{g}$ with modulus 1 eigenvalues that keeps the bilinear form invariant. This automorphism extends to two automorphisms of the Lie conformal superalgebra $R^{super}$, denoted by $\sigma$ and $\tau$, as follows. Both fix $\mathcal{K}$ and $\overline{\mathcal{K}}$ and commute with $T$, both act on $1\otimes\mathfrak{g}$ as $1\otimes\sigma$, and $\sigma$ (resp. $\tau$) acts on $1\otimes\overline{\mathfrak{g}}$ as $1\otimes\sigma$ (resp. $-1\otimes\sigma$). We therefore obtain two automorphisms of $V(R^{super})$, also denoted by $\sigma$ and $\tau$. Clearly $\sigma$ and $\tau$ stabilize

$$:(\mathcal{K}-k|0\rangle)V(R^{super}): + :(\overline{\mathcal{K}}-|0\rangle)V(R^{super}):,$$

so we obtain automorphisms of $V^{k,1}(R^{super})$, also denoted by $\sigma$ and $\tau$.

Denote by $\overline{\tau}$ the automorphism $\tau$ restricted to $R^{Cl}(\overline{\mathfrak{g}}) = \mathbb{C}[T]\otimes\overline{\mathfrak{g}}+\mathbb{C}\overline{\mathcal{K}}$. As above, we can extend this automorphism to $F(\overline{\mathfrak{g}})$, also denoted by $\overline{\tau}$.

Observe that $\widetilde{\sigma(x)} = \tau(\tilde{x})$ for $x\in\mathfrak{g}$. Indeed,

$$\sigma(x)-\frac{1}{2}\sum_i :\overline{[\sigma(x),x_i]}\overline{x}_i: = \tau(x)-\frac{1}{2}\sigma(\sum_i :\overline{[x,\sigma^{-1}(x_i)]}\,\overline{\sigma^{-1}(x_i)}): = \tau(\tilde{x}). \tag{3.17}$$

**Remark 3.1.** It follows from (3.17) that the isomorphism $V^{k+g,1}(R^{super}) \to V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$ intertwines $\tau$ and $\sigma \otimes \overline{\tau}$. Thus, if $M$ is a level $k$ highest weight $L'(\mathfrak{g}, \sigma)$-module, then $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is a $\sigma \otimes \overline{\tau}$-twisted representation of $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$ hence a $\tau$-twisted representation of $V^{k+g,1}(R^{super})$. In particular, if $b \in \mathfrak{g}$, then $\widetilde{b}$ acts only on the first factor of $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ whereas $\overline{b}$ acts only on the second factor.

Next let $\mathfrak{a}$ be a reductive $\sigma$-invariant subalgebra of $\mathfrak{g}$, such that $(\cdot, \cdot)$ remains nondegenerate when restricted to $\mathfrak{a}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$. If $x \in \mathfrak{g}$ we write $x = x_{\mathfrak{a}} + x_{\mathfrak{p}}$ for the orthogonal decomposition of $x$. We fix once and for all an orthonormal basis $\{a_i\}$ of $\mathfrak{a}$ and an orthonormal basis $\{b_i\}$ of $\mathfrak{p}$.

Let $Cas_{\mathfrak{a}}$ be the Casimir of $\mathfrak{a}$ with respect to $(\cdot, \cdot)_{|\mathfrak{a}}$. Write $\mathfrak{a} = \sum_S \mathfrak{a}_S$ for the eigenspace decomposition of $\mathfrak{a}$ under the action of $Cas_{\mathfrak{a}}$, and let $2g_S$ be the eigenvalue relative to $\mathfrak{a}_S$. In particular we let $\mathfrak{a}_0$ be the center of $\mathfrak{a}$, while $\mathfrak{a}_S$ is semisimple for $S > 0$.

We construct the Lie conformal algebra $Cur(\mathfrak{a}_S)$ and, given $k \in \mathbb{C}$, the corresponding vertex algebra $V^k(\mathfrak{a}_S)$ using the form $(\cdot, \cdot)$ restricted to $\mathfrak{a}_S$, so that, if $x, y \in \mathfrak{a}_S$, then $[x_\lambda y] = [x, y] + \lambda(x, y)K_S$. We abuse slightly of notation by letting $Cur(\mathfrak{a}) = (\mathbb{C}[T] \otimes \mathfrak{a}) \oplus (\oplus_S \mathbb{C}K_S)$.

The super affine conformal algebra $R^{super}(\mathfrak{a})$ corresponding to $\mathfrak{a}$ embeds naturally in $R^{super}$, thus we have an embedding of $V^{k+g,1}(R^{super}(\mathfrak{a}))$ in $V^{k+g,1}(R^{super})$. In particular $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ turns into a representation of $V^{k+g,1}(R^{super}(\mathfrak{a}))$ by restriction. Set, for $x \in \mathfrak{a}$,

$$(\tilde{x})_{\mathfrak{a}} = x - \frac{1}{2} \sum_i :\overline{[x, a_i]}\overline{a}_i: .$$

Since $ad(Cas_{\mathfrak{a}})_{|\mathfrak{a}_S} = 2g_S I_{\mathfrak{a}_S}$, applying Proposition 2.1 to $R^{super}(\mathfrak{a})$, we have that $x \mapsto (\tilde{x})_{\mathfrak{a}}$, $\overline{x} \mapsto \overline{x}$ induces an isomorphism $(\otimes_S V^{k+g-g_S}(\mathfrak{a}_S)) \otimes F^{\overline{\tau}}(\overline{\mathfrak{a}}) \to V^{k+g,1}(R^{super}(\mathfrak{a}))$ that intertwines $\sigma \otimes \overline{\tau}$ with $\tau$.

It follows that we can look upon $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ as a $\sigma \otimes \overline{\tau}$-twisted representation of $(\otimes_S V^{k+g-g_S}(\mathfrak{a}_S)) \otimes F^{\overline{\tau}}(\overline{\mathfrak{a}})$.

In order to understand this representation we write $N = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ as $(M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})) \otimes F^{\overline{\tau}}(\overline{\mathfrak{a}})$. For $x \in \mathfrak{a}$ set

$$\theta(x) = (\tilde{x})_{\mathfrak{a}} - \tilde{x}. \tag{3.18}$$

Then $x$ acts on $N$ via $Y^N((\tilde{x})_{\mathfrak{a}}, z) = Y^N(\tilde{x}, z) + Y^N(\theta(x), z) = Y^M(x, z) \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{g}})} + I_M \otimes Y^{F^{\overline{\tau}}(\overline{\mathfrak{g}})}(\theta(x), z)$. Since

$$\theta(x) = \frac{1}{2} \sum_i :\overline{[x, b_i]}\overline{b}^i:, \tag{3.19}$$

we have that $\theta(x) \in F(\overline{\mathfrak{p}})$, and in turn $Y^{F^{\overline{\tau}}(\overline{\mathfrak{g}})}(\theta(x), z) = Y^{F^{\overline{\tau}}(\overline{\mathfrak{p}})}(\theta(x), z) \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{a}})}$. Thus, as a representation of $\otimes_S V^{k+g-g_S}(\mathfrak{a}_S) \otimes F^{\overline{\tau}}(\overline{\mathfrak{a}})$, $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is $(M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})) \otimes F^{\overline{\tau}}(\overline{\mathfrak{a}})$, where the representation of $V^{k+g-g_S}(\mathfrak{a}_S)$ on $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ is the one induced by the field module of $Cur(\mathfrak{a})$ defined by

$$Y^M(x, z) \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{p}})} + I_M \otimes Y^{F^{\overline{\tau}}(\overline{\mathfrak{p}})}(\theta(x), z). \tag{3.20}$$

Recall that $\Delta_0^+$ is a subset of positive roots for the set of $\mathfrak{h}_0$-roots of $\mathfrak{g}^{\overline{0}}$. Let $\mathfrak{b} = \mathfrak{h}_0 \oplus \mathfrak{n}$ be the corresponding Borel subalgebra. Fix a Cartan subalgebra $\mathfrak{h}_\mathfrak{a}$ of $\mathfrak{a}^{\overline{0}} = \mathfrak{a} \cap \mathfrak{g}^{\overline{0}}$. We can assume that $\mathfrak{h}_\mathfrak{a} \subset \mathfrak{h}_0$, so that $\mathfrak{h}_0 = \mathfrak{h}_\mathfrak{a} \oplus \mathfrak{h}_\mathfrak{p}$ is the orthogonal decomposition of $\mathfrak{h}_0$. Furthermore, as shown in § 1.1 of [13], we can assume that $\mathfrak{n} = \mathfrak{n} \cap \mathfrak{a}^{\overline{0}} \oplus \mathfrak{n} \cap (\mathfrak{p} \cap \mathfrak{g}^{\overline{0}})$ and that, if $\mathfrak{n}_\mathfrak{a} = \mathfrak{n} \cap \mathfrak{a}^{\overline{0}}$, then $\mathfrak{h}_\mathfrak{a} \oplus \mathfrak{n}_\mathfrak{a}$ is a Borel subalgebra of $\mathfrak{a}^{\overline{0}}$. Let $\Delta_\mathfrak{a}^+$ be the corresponding subset of positive roots in the set $\Delta_\mathfrak{a}$ of $\mathfrak{h}_\mathfrak{a}$-roots of $\mathfrak{a}^{\overline{0}}$. Set $\mathfrak{n}_- = \sum\limits_{\alpha \in -\Delta_0^+} \mathfrak{g}^{\overline{0}}_\alpha$. The same argument used for $\mathfrak{n}$ shows that $\mathfrak{n}_- = \mathfrak{n}_- \cap \mathfrak{a}^{\overline{0}} \oplus \mathfrak{n}_- \cap (\mathfrak{p} \cap \mathfrak{g}^{\overline{0}})$. We define $L'(\mathfrak{a}, \sigma) = \sum_{j \in \mathbb{R}} t^j \otimes \mathfrak{a}^j \oplus (\sum_S \mathbb{C} K_S)$. This is a Lie algebra with bracket defined by

$$[t^i \otimes a, t^j \otimes b] = t^{i+j} \otimes [a, b] + \delta_{i,j} i(a, b) K_S$$

for $a \in \mathfrak{a}_S$, the elements $K_S$ being central.

Set $\mathfrak{n}'_\mathfrak{a} = \mathfrak{n}' \cap L'(\mathfrak{a}, \sigma)$. If $\mu \in (\mathfrak{h}_\mathfrak{a} \oplus (\sum_S \mathbb{C} K_S))^*$, we say that a $L'(\mathfrak{a}, \sigma)$-module $M$ is a highest weight module of weight $\mu$ if there is a nonzero vector $v_\mu \in M$ such that

$$\mathfrak{n}'_\mathfrak{a}(v_\mu) = 0, \;\; hv_\mu = \mu(h) v_\mu \text{ for } h \in \mathfrak{h}_\mathfrak{a} \oplus (\sum_S \mathbb{C} K_S), \;\; U(L'(\mathfrak{a}, \sigma)) v_\mu = M. \tag{3.21}$$

If $M$ is a highest weight module for $L'(\mathfrak{g}, \sigma)$ with highest weight $\Lambda$ and $k = \Lambda(K)$, then $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ is a representation of $\otimes_S V^{k+g-g_S}(\mathfrak{a}_S)$, thus we can regard $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ as a $L'(\mathfrak{a}, \sigma)$-module. In particular, by letting $M$ be the trivial representation of $L'(\mathfrak{g}, \sigma)$, we have an action of $L'(\mathfrak{a}, \sigma)$ on $F^{\overline{\tau}}(\overline{\mathfrak{p}})$.

Let $\Lambda_0^S$ be the element of $(\mathfrak{h}_\mathfrak{a} + \sum_S \mathbb{C} K_S)^*$ defined by

$$\Lambda_0^S(\mathfrak{h}_\mathfrak{a}) = 0, \quad \Lambda_0^S(K_T) = \delta_{ST}.$$

Define moreover

$$\rho_{\mathfrak{a}j} = \frac{1}{2} \sum_{\alpha \in (\Delta_{\overline{j}})_{|\mathfrak{h}_\mathfrak{a}}} \dim(\mathfrak{a}^{\overline{j}})_\alpha \alpha, \quad \rho_{\mathfrak{a}\sigma} = \sum_{0 \le j < \frac{1}{2}} (1 - 2j) \rho_{\mathfrak{a}j}. \tag{3.22}$$

**Lemma 3.4.** *Let $M$ be a level $k$ highest weight module for $L'(\mathfrak{g},\sigma)$ with highest weight $\Lambda$. Set $M' = U(L'(\mathfrak{a},\sigma))(v_\Lambda \otimes 1)$. Then $M'$ is a highest weight $L'(\mathfrak{a},\sigma)$-module with highest weight*

$$\mu = (\Lambda + \rho_\sigma)_{|\mathfrak{h}_\mathfrak{a}} - \rho_{\mathfrak{a}\sigma} + \sum_S (k + g - g_S)\Lambda_0^S. \tag{3.23}$$

*Proof.* If $x \in \mathfrak{a}^j$ (with $-\frac{1}{2} \le j < \frac{1}{2}$) then, by (3.20), $t^n \otimes x$ acts via $x^M_{(n)} \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{p}})} + I_M \otimes \theta(x)^{\overline{\tau}}_{(n)}$. If $n > 0$ then $x^M_{(n)}(v_\Lambda) = 0$. Moreover $\theta(x)^{\overline{\tau}}_{(n)} = \frac{1}{2}\sum_i :\overline{[x,b_i]}\overline{b}^i:^{\overline{\tau}}_{(n)}$, hence, using (3.4),

$$\begin{aligned}
\theta(x)^{\overline{\tau}}_{(n)}(1) &= \frac{1}{2}\sum_{i:j+s_i>0} \overline{[x,b_i]}^{\overline{\tau}}_{(j+s_i-\frac{1}{2})}(\overline{b}^i)^{\overline{\tau}}_{(-j-s_i-\frac{1}{2}+n)}(1)\\
&\quad - \frac{1}{2}\sum_{i:j=s_i=-\frac{1}{2}} (\overline{b}^i)^{\overline{\tau}}_{(-\frac{1}{2}+n)}\overline{[x,b_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(1)\\
&= 0.
\end{aligned}$$

If $n = 0$ and $x \in \mathfrak{n}_\mathfrak{a}$ then, since $\mathfrak{n}_\mathfrak{a} \subset \mathfrak{n}$, $x^M_{(0)}(v_\Lambda) = 0$. Moreover

$$\begin{aligned}
\theta(x)^{\overline{\tau}}_{(0)}(1) &= \frac{1}{2}\sum_{i:s_i\ge 0} \overline{[x,b_i]}^{\overline{\tau}}_{(s_i-\frac{1}{2})}(\overline{b}^i)^{\overline{\tau}}_{(-s_i-\frac{1}{2})}(1)\\
&= \frac{1}{2}\sum_{i:s_i=0} \overline{[x,b_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{b}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(1)\\
&\quad + \frac{1}{2}\sum_{i:s_i>0} ([x,b_i],b^i).
\end{aligned}$$

Since $x \in \mathfrak{n}$ and, if $s_i > 0$, $[b_i, b^i] \in \mathfrak{h}_0$, we have that $\sum_{i,s_i>0}([x,b_i],b^i) = \sum_{i,s_i>0}(x,[b_i,b^i]) = 0$. We choose a maximal isotropic space $\mathfrak{h}_\mathfrak{p}^+$ of $\mathfrak{h}_\mathfrak{p}$, so that $(\mathfrak{n}\cap\mathfrak{p}) \oplus \mathfrak{h}_\mathfrak{p}^+$ is a maximal isotropic space in $\mathfrak{g}^{\overline{0}} \cap \mathfrak{p}$.

We can choose the basis $\{b_i\}$ as the union of a basis $\{x_i\}$ of $\mathfrak{n}\cap\mathfrak{p}$ with an orthonormal basis $\{h_i\}$ of $\mathfrak{h}_\mathfrak{p}$ and a basis $\{y_i\}$ of $\mathfrak{n}_-\cap\mathfrak{p}$. Set $\{x^i\}$ (resp. $\{y^i\}$) be the basis of $\mathfrak{n}_-\cap\mathfrak{p}$ (resp. $\mathfrak{n}\cap\mathfrak{p}$) dual to $\{x_i\}$ (resp. $\{y_i\}$). Then

$$\begin{aligned}
\sum_{i:s_i=0} \overline{[x,b_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{b}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(1) &= \sum_i \overline{[x,x_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{x}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(1)\\
&\quad + \sum_i \overline{[x,h_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{h}_i)^{\overline{\tau}}_{(-\frac{1}{2})}(1)\\
&\quad + \sum_i \overline{[x,y_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{y}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(1).
\end{aligned}$$

Since $x \in \mathfrak{n}$, then $[x,h] \in \mathfrak{n}\cap\mathfrak{p}$ and $[x,x_i] \in \mathfrak{n}\cap\mathfrak{p}$, thus

$$\sum_{i:s_i=0} \overline{[x,b_i]}^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{b}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(1) = \sum_i (x,[x_i,x^i]) + \sum_i (x,[h_i,h_i]) = 0.$$

It remains to compute the highest weight $\mu$. If $h \in \mathfrak{h}_{\mathfrak{a}}$, then

$$(h)^N_{(0)} \cdot (v_\Lambda \otimes 1) = hv_\Lambda \otimes 1 + v_\Lambda \otimes \theta(h)^{\overline{\tau}}_{(0)} \cdot 1 = \Lambda(h)v_\Lambda \otimes 1 + v_\Lambda \otimes \theta(h)^{\overline{\tau}}_{(0)} \cdot 1.$$

Now

$$\theta(h)^{\overline{\tau}}_{(0)} \cdot 1 = \sum_i :\overline{[h,b_i]}\overline{b}^i:^{\overline{\tau}}_{(0)} \cdot 1.$$

Applying (3.4) we find that

$$Y^{\overline{\tau}}(\theta(h),z) = \frac{1}{2}\sum_i :Y^{\overline{\tau}}(\overline{[h,b_i]},z)Y^{\overline{\tau}}(\overline{b}^i,z): - \sum_{\substack{j>0\\ \alpha\in(\Delta_{\overline{j}})_{|\mathfrak{h}_{\mathfrak{a}}}}} j\alpha(h)\dim(\mathfrak{g}^j\cap\mathfrak{p})_\alpha z^{-1}.$$

Writing out explicitly the normal order in the r.h.s of the previous equation, we get:

$$\begin{aligned}
&\theta(h)^{\overline{\tau}}_{(0)} \cdot 1 = \\
&\frac{1}{2}\sum_i \overline{[h,b_i]}^{\overline{\tau}}_{s_i-\frac{1}{2}}(\overline{b}^i)^{\overline{\tau}}_{-s_i-\frac{1}{2}} \cdot 1 - \sum_{j>0} 2j(\rho_j - \rho_{\mathfrak{a}j})(h) = \\
&\frac{1}{2}\sum_{i:s_i=0} \overline{[h,b_i]}^{\overline{\tau}}_{-\frac{1}{2}}(\overline{b}^i)^{\overline{\tau}}_{-\frac{1}{2}} \cdot 1 + \frac{1}{2}\sum_{i:s_i>0}([h,b_i],b^i) - \sum_{j>0} 2j(\rho_j - \rho_{\mathfrak{a}j})(h) \\
&\frac{1}{2}\sum_{i:s_i=0} \overline{[h,b_i]}^{\overline{\tau}}_{-\frac{1}{2}}(\overline{b}^i)^{\overline{\tau}}_{-\frac{1}{2}} \cdot 1 + \sum_{j>0}(\rho_j - \rho_{\mathfrak{a}j})(h) - \sum_{j>0} 2j(\rho_j - \rho_{\mathfrak{a}j})(h).
\end{aligned}$$

Choosing bases $\{x_i\}$ in $\mathfrak{n}\cap\mathfrak{p}$, $\{y_i\}$ in $\mathfrak{n}^-\cap\mathfrak{p}$ and $\{h_i\}$ in $\mathfrak{h}_{\mathfrak{p}}$ as above we have

$$\frac{1}{2}\sum_{i:s_i=0} \overline{[h,b_i]}^{\overline{\tau}}_{-\frac{1}{2}}(\overline{b}^i)^{\overline{\tau}}_{-\frac{1}{2}} \cdot 1 = \frac{1}{2}\sum_{i:s_i=0} \overline{[h,x_i]}^{\overline{\tau}}_{-\frac{1}{2}}(\overline{x}^i)^{\overline{\tau}}_{-\frac{1}{2}} \cdot 1 = (\rho_0 - \rho_{\mathfrak{a}0})(h).$$

The final outcome is that

$$\theta(h)^{\overline{\tau}}_{(0)} \cdot 1 = (\rho_\sigma - \rho_{\mathfrak{a}\sigma})(h).$$

Since we are looking at $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ as a representation of $\otimes_S V^{k+g-g_S}(\mathfrak{a}_S)$, then $K_S$ acts as $(k+g-g_S)I$. $\square$

Similarly to what we have done with $L'(\mathfrak{g},\sigma)$, we define $\widehat{L}(\mathfrak{a},\sigma)$ by extending $L'(\mathfrak{a},\sigma)$ with $d_\mathfrak{a}$ and letting $d_\mathfrak{a}$ and $K_S$ commute for all $S$. We wish to extend the action of $L'(\mathfrak{a},\sigma)$ on $F^{\overline{\tau}}(\overline{\mathfrak{p}})$ to $\widehat{L}(\mathfrak{a},\sigma)$. In order to do this we need the following computation. Recall from (3.14) the definition of the element $L^A$ for the vector superspace $A$.

**Lemma 3.5.** *In $V^{k,1}(R^{super})$,*

$$[\theta(x)_\lambda L^{\overline{\mathfrak{p}}}] = -\lambda\theta(x).$$

*Consequently, due to* (3.1)*, for any $a \in \mathbb{C}$, we can extend the action of $L'(\mathfrak{a},\sigma)$ on $F^{\overline{\tau}}(\overline{\mathfrak{p}})$ to $\widehat{L}(\mathfrak{a},\sigma)$ by letting $d_\mathfrak{a}$ act as $(L^{\overline{\mathfrak{p}}})^{\overline{\tau}}_{(1)} + aI$.*

*Proof.* By (2.9) and Wick formula (2.5), we have

$$\begin{aligned}
\frac{1}{2}\sum_i[\theta(x)_\lambda : T(\overline{b}_i)\overline{b}_i :] &= \frac{1}{2}\sum_i[(\tilde{x})_{\mathfrak{a}\lambda} : T(\overline{b}_i)\overline{b}_i :] - \frac{1}{2}\sum_i[\tilde{x}_\lambda : T(\overline{b}_i)\overline{b}_i :] \\
&= \frac{1}{2}\sum_i[(\tilde{x})_{\mathfrak{a}\lambda} : T(\overline{b}_i)\overline{b}_i :] \\
&= \frac{1}{2}\sum_i(: [(\tilde{x})_{\mathfrak{a}\lambda} T(\overline{b}_i)]\overline{b}_i : + : T(\overline{b}_i)[(\tilde{x})_{\mathfrak{a}\lambda} b_i] :) \\
&+ \frac{1}{2}\sum_i\int_0^\lambda [[(\tilde{x})_{\mathfrak{a}\lambda} T(\overline{b}_i)]_\mu \overline{b}_i] d\mu.
\end{aligned}$$

As in the proof of Proposition 2.1 it can be computed easily that, if $y \in \mathfrak{p}$ and $x \in \mathfrak{a}$, then $[(\tilde{x})_{\mathfrak{a}\lambda}\overline{y}] = \overline{[x,y]}$. By sesquilinearity of the $\lambda$-bracket we have then $[(\tilde{x})_{\mathfrak{a}\lambda} T(\overline{b}_i)] = T([(\tilde{x})_{\mathfrak{a}\lambda}\overline{b}_i]) + \lambda[(\tilde{x})_{\mathfrak{a}\lambda}\overline{b}_i] = T(\overline{[x,b_i]}) + \lambda\overline{[x,b_i]}$, hence we can write

$$\begin{aligned}
\frac{1}{2}\sum_i[\theta(x)_\lambda : T(\overline{b}_i)\overline{b}_i :] &= \frac{1}{2}\sum_i(: T(\overline{[x,b_i]})\overline{b}_i : + \lambda : \overline{[x,b_i]}\overline{b}_i : + : T(\overline{b}_i)\overline{[x,b_i]} :) \\
&+ \frac{1}{2}\sum_i\int_0^\lambda [(T(\overline{[x,b_i]}) + \lambda\overline{[x,b_i]})_\mu \overline{b}_i] d\mu \\
&= \lambda\theta(x) + \frac{1}{2}\sum_i\int_0^\lambda (-\lambda([x,b_i],b_i) + \lambda([x,b_i],b_i) d\mu \\
&= \lambda\theta(x).
\end{aligned}$$

□

# 4 Dirac operators

The affine Dirac operator was introduced by Kac and Todorov in [10]. It is the following odd element of $V^{k+g,1}(R^{super})$:

$$G_{\mathfrak{g}} = \sum_i :x_i\overline{x}^i: + \tfrac{1}{3}\sum_{i,j} :\overline{[x_i,x_j]}\overline{x}^i\overline{x}^j: . \tag{4.1}$$

Here $\{x_i\}$, $\{x^i\}$ is a pair of dual bases of $\mathfrak{g}$ w.r.t. the invariant form $(\ ,\ )$. Then, choosing $x_i$ as the eigenvectors of $\sigma$, say $\sigma(x_i) = a_i x_i$, we see that $\tau(\overline{x}_i) = -a_i\overline{x}_i$, $\tau(\overline{x}^i) = -a_i^{-1}\overline{x}^i$, hence

$$\tau(G_{\mathfrak{g}}) = -G_{\mathfrak{g}}. \tag{4.2}$$

The element $G_{\mathfrak{g}}$ has the following properties:

$$[a_\lambda G_{\mathfrak{g}}] = \lambda(k+g)\overline{a}, \tag{4.3}$$
$$[\overline{a}_\lambda G_{\mathfrak{g}}] = a. \tag{4.4}$$

Note that

$$G_{\mathfrak{g}} = \sum_i :\tilde{x}_i\overline{x}^i: - \tfrac{1}{6}\sum_{i,j} :\overline{[x_i,x_j]}\overline{x}^i\overline{x}^j: . \tag{4.5}$$

In Section 9.1 we will show (cf. [1]) that

$$[G_{\mathfrak{g}\lambda}G_{\mathfrak{g}}] = \sum_i :\tilde{x}_i\tilde{x}^i: + (k+g)\sum_i :T(\overline{x}_i)\overline{x}^i: + \frac{\lambda^2}{2}(k+\frac{g}{3})\dim\mathfrak{g}. \tag{4.6}$$

Identifying $V^{k+g,1}(R^{super})$ with $V^k(\mathfrak{g})\otimes F(\overline{\mathfrak{g}})$ we have that (4.5) and (9.1) can be rewritten as

$$G_{\mathfrak{g}} = \sum_i x_i\otimes\overline{x}^i - \tfrac{1}{6}\sum_{i,j}|0\rangle\otimes :\overline{[x_i,x_j]}\overline{x}^i\overline{x}^j: \tag{4.7}$$

and

$$[G_{\mathfrak{g}\lambda}G_{\mathfrak{g}}] = 2(L^{\mathfrak{g}}\otimes|0\rangle) - 2(k+g)(|0\rangle\otimes L^{\overline{\mathfrak{g}}}) + \frac{\lambda^2}{2}(k+\frac{g}{3})\dim\mathfrak{g}. \tag{4.8}$$

where $L^{\mathfrak{g}}$ is defined in (3.7) and $L^{\overline{\mathfrak{g}}}$ is defined in (3.14). Note that if $L_{\mathfrak{g}}$ is as in (1.5), then we have

$$L_{\mathfrak{g}} = 2(L^{\mathfrak{g}}\otimes|0\rangle) - 2(k+g)(|0\rangle\otimes L^{\overline{\mathfrak{g}}}).$$

We observe that $G_{\mathfrak{g}} \in V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$, so, fixing a restricted module $M$ for $L'(\mathfrak{g}, \sigma)$ of level $k$ and setting $N = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$, we can consider the twisted quantum field

$$Y^N(G_{\mathfrak{g}}, z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} G^N_{(n)} z^{-n-1} = \sum_{n \in \mathbb{Z}} G^N_{(\frac{1}{2}+n)} z^{-n-\frac{3}{2}}.$$

(Recall from (4.2) that $\overline{\tau}(G_{\mathfrak{g}}) = -G_{\mathfrak{g}}$.) Let $G^N_n = G^N_{(\frac{1}{2}+n)}$. We want to calculate $(G^N_0)^2$. Using (4.8) we have

$$(G^N_0)^2 = \frac{1}{2}[G^N_0, G^N_0] = \frac{1}{2}[G^N_{(\frac{1}{2})}, G^N_{(\frac{1}{2})}] = \tfrac{1}{2}(G_{\mathfrak{g}(0)} G_{\mathfrak{g}})^N_{(1)} - \tfrac{1}{16}(k + \frac{g}{3})(\dim \mathfrak{g})\, I_N. \tag{4.9}$$

Combining (4.9) and (4.8) with $\lambda = 0$, we obtain

$$(G^N_0)^2 = (L^{\mathfrak{g}})^M_{(1)} \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{g}})} - (k+g) I_M \otimes (L^{\overline{\mathfrak{g}}})^{\overline{\tau}}_{(1)} - \tfrac{1}{16}(k + \frac{g}{3})(\dim \mathfrak{g})\, I_N. \tag{4.10}$$

We are interested in calculating $G^N_0(v_\Lambda \otimes 1)$, $v_\Lambda$ being a highest weight vector of a $L'(\mathfrak{g}, \sigma)$-module $M$ with highest weight $\Lambda$. From (4.7) we know that $G^N_0$ splits as the sum of a quadratic and a cubic term. We shall calculate the action of these two sums separately. We assume that $x_i \in \mathfrak{g}^{\overline{s}_i}$ where $-\frac{1}{2} \le s_i < \frac{1}{2}$, so that $x^i \in \mathfrak{g}^{-\overline{s}_i}$. Writing $Y^{\overline{\tau}}$ for $Y^{F^{\overline{\tau}}(\overline{\mathfrak{g}})}$ and, if $a \in F(\overline{\mathfrak{g}})$, $a^{\bar{\tau}}_{(r)}$ for $a^{F^{\overline{\tau}}(\overline{\mathfrak{g}})}_{(r)}$, we have

$$\begin{aligned} &Y^{\overline{\tau}}(\sum_{i,j} :\overline{[x_i, x_j]}\overline{x}^i \overline{x}^j:, z) = \\ &\sum_{i,j} : Y^{\overline{\tau}}(\overline{[x_i, x_j]}, z) Y^{\overline{\tau}}(\overline{x}^i, z) Y^{\overline{\tau}}(\overline{x}^j, z) : + 3 \sum_i (s_i + \frac{1}{2}) Y^{\overline{\tau}}(\overline{[x_i, x^i]}, z) z^{-1}. \end{aligned} \tag{4.11}$$

This equality follows by repeated applications of (3.4). We now simplify the second summand of the right hand side. We already observed that

$$\sum_{s_i = t} [x_i, x^i] = - \sum_{s_i = -t} [x_i, x^i]$$

hence in particular $\sum_{s_i=0} [x_i, x^i] = 0$. Thus

$$\sum_i (s_i + \frac{1}{2})[x_i, x^i] = \sum_{0 < s_i < \frac{1}{2}} 2 s_i [x_i, x^i].$$

If we single out the coefficient of $\sum_{i,j} : Y^{\overline{\tau}}(\overline{[x_i, x_j]}, z)Y^{\overline{\tau}}(\overline{x}^i, z)Y^{\overline{\tau}}(\overline{x}^j, z) :$ corresponding to $z^{-\frac{1}{2}-1}$ and apply it to $v_\Lambda \otimes 1$ (indeed to 1), we have

$$\sum_{i,j} : Y^{\overline{\tau}}(\overline{[x_i, x_j]_{\mathfrak{p}}}, z)Y^{\overline{\tau}}(\overline{x}^i, z)Y^{\overline{\tau}}(\overline{x}^j, z) :_{(\frac{1}{2})} (1) =$$

$$\sum_{i,j}( \sum_{s_i=s_j=0} (\overline{[x_i, x_j]})^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{x}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{x}^j)^{\overline{\tau}}_{(-\frac{1}{2})})(1) - 3\sum_{s_j>0}(\overline{[x_j, x^j]})^{\overline{\tau}}_{(-\frac{1}{2})}(1). \quad (4.12)$$

The first summand in (4.12) is the cubic term in Kostant's Dirac operator for $\mathfrak{g}^{\overline{0}}$. In Kostant [13] it is proven that

$$\sum_{i,j}( \sum_{s_i=s_j=0} (\overline{[x_i, x_j]})^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{x}^i)^{\overline{\tau}}_{(-\frac{1}{2})}(\overline{x}^j)^{\overline{\tau}}_{(-\frac{1}{2})})(1) = -6(\overline{h}_{\rho_0})^{\overline{\tau}}_{(-\frac{1}{2})}(1).$$

With easy calculations one proves that

$$\left(\sum_i x_i \otimes \overline{x}^i\right)^N_{(\frac{1}{2})} (v_\Lambda \otimes 1) = v_\Lambda \otimes (h_{\overline{\Lambda}})^{\overline{\tau}}_{(-\frac{1}{2})} \cdot 1. \quad (4.13)$$

Now we can complete the proof of

**Proposition 4.1.** *Let $\rho_\sigma$ be as in* (3.6)*. Then*

$$G_0^N(v_\Lambda \otimes 1) = v_\Lambda \otimes (\overline{h}_{\overline{\Lambda}+\rho_\sigma})^{\overline{\tau}}_{(-\frac{1}{2})} \cdot 1. \quad (4.14)$$

*Proof.* Collecting all the contributions (4.11),(4.12), and (4.13), we find that

$$G_0^N(v_\Lambda \otimes 1) = v_\Lambda \otimes \left(\overline{h}_{\overline{\Lambda}+\rho_0} + \frac{1}{2} \sum_{j:\frac{1}{2}>s_j>0} (1 - 2s_j)\overline{[x_j, x^j]}\right)^{\overline{\tau}}_{(-\frac{1}{2})} \cdot 1. \quad (4.15)$$

Now, if $\{v_i\}$ is a basis of $\mathfrak{g}^{\overline{t}}$, then $\sum\limits_{s_i=t}(1 - 2s_i)[v_i, v^i]$ is independent of the choice of the basis. It follows that

$$\sum_{i:s_i=t} (1 - 2s_i)[x_i, x^i] = \sum_{i:s_i=t} \sum_{\alpha\in\Delta_{\overline{t}}} (1 - 2s_i)[x_{\alpha i}, x^i_\alpha] \quad (4.16)$$

where $\{x_{\alpha i}\}$ is a basis of $\mathfrak{g}^{\overline{t}}_\alpha$. Since $[x_{\alpha i}, x^i_\alpha] = h_\alpha$, we have that the l.h.s. of (4.16) equals $2(1-2t)h_{\rho_t}$. Summing over $t$ and substituting in (4.15) we get (4.14). □

Recall from (3.8) the definition of $z(\mathfrak{g}, \sigma)$. We have

**Proposition 4.2.**

1. $(L^{\overline{\mathfrak{g}}})^{\overline{\tau}}_{(1)} \cdot 1 = z(\mathfrak{g},\sigma) - \frac{1}{16}\dim\mathfrak{g}$.

2. *If $M$ is a highest weight module of $L'(\mathfrak{g},\sigma)$ with highest weight $\Lambda$, then*

$$(G_0^N)^2(v_\Lambda \otimes 1) = \frac{1}{2}\left((\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda}) + \frac{g}{12}\dim\mathfrak{g} - 2gz(\mathfrak{g},\sigma)\right)(v_\Lambda\otimes 1).$$

*Proof.* If $M_0$ is a highest weight module with highest weight $-\rho_\sigma + k\Lambda_0$ then, by Proposition 4.1, $G_0^N(v_{-\rho_\sigma+k\Lambda_0}\otimes 1) = 0$. Applying (4.10) and Lemma 3.2, we find that

$$0 = (-\frac{1}{2}\|\rho_\sigma\|^2 + kz(\mathfrak{g},\sigma) - \tfrac{1}{16}(k+\frac{g}{3})\dim\mathfrak{g})(v_\Lambda\otimes 1) - (k+g)(v_\Lambda\otimes (L^{\overline{\mathfrak{g}}})^{\overline{\tau}}_{(1)}\cdot 1).$$

Since this equality holds for any $k$, the coefficient of $k$ must vanish. This implies the first claim of the proposition.

Again by (4.10) and Lemma 3.2,

$$\begin{aligned}(G_0^N)^2(v_\Lambda\otimes 1) = (&\frac{1}{2}(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda}) + kz(\mathfrak{g},\sigma) - \tfrac{1}{16}(k+\frac{g}{3})\dim\mathfrak{g})(v_\Lambda\otimes 1)\\ &- (k+g)(v_\Lambda\otimes(L^{\overline{\mathfrak{g}}})^{\overline{\tau}}_{(1)}\cdot 1).\end{aligned}$$

Using the first equality we get the second claim. □

We now turn to the study of the relative Dirac operator. Fix a subalgebra $\mathfrak{a}$ as in § 3.3 and consider $G_{\mathfrak{a}} \in V^{k+g,1}(R^{super}(\mathfrak{a})) \subset V^{k+g,1}(R^{super})$.

Set $G_{\mathfrak{g},\mathfrak{a}} = G_{\mathfrak{g}} - G_{\mathfrak{a}}$. By (4.3) and (4.4),

$$[{G_{\mathfrak{g},\mathfrak{a}}}_\lambda G_{\mathfrak{g},\mathfrak{a}}] = [{G_{\mathfrak{g}}}_\lambda G_{\mathfrak{g}}] - [{G_{\mathfrak{a}}}_\lambda G_{\mathfrak{a}}].$$

In particular

$$((G_{\mathfrak{g},\mathfrak{a}})_0^N)^2 = (G_0^N)^2 - ((G_{\mathfrak{a}})_0^N)^2, \tag{4.17}$$

so, by (4.10),

$$\begin{aligned}((G_{\mathfrak{g},\mathfrak{a}})_0^N)^2 = &(L^{\mathfrak{g}})^M_{(1)} \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{g}})} - (k+g)I_M\otimes(L^{\overline{\mathfrak{g}}} - L^{\overline{\mathfrak{a}}})^{\overline{\tau}}_{(1)}\\ &- (L^{\mathfrak{a}})^{M\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})}_{(1)} \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{a}})}\\ &- \tfrac{1}{16}\left((k+\frac{g}{3})\dim\mathfrak{g} - \sum_S(k+g-\frac{2g_S}{3})\dim\mathfrak{a}_S\right)I_N.\end{aligned} \tag{4.18}$$

**Proposition 4.3.** *If* $(\overline{\Lambda}+\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}=0$ *then* $(G_{\mathfrak{g},\mathfrak{a}})_0^N(v_\Lambda\otimes 1)=0$ *and, if* $\mu$ *is as in* (3.23),

$$(\overline{\mu}+2\rho_{\mathfrak{a}\sigma},\overline{\mu})-(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda})\\ =\frac{g}{12}\dim\mathfrak{g}-2gz(\mathfrak{g},\sigma)-\sum_S(\frac{g_S}{12}\dim\mathfrak{a}_S-2g_Sz(\mathfrak{a}_S,\sigma)). \tag{4.19}$$

*Proof.* Since $N=M\otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})=(M\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}))\otimes F^{\overline{\tau}}(\overline{\mathfrak{a}})$, applying Proposition 4.1 to $G_\mathfrak{a}$ and using Lemma 3.4, we find that

$$(G_{\mathfrak{g},\mathfrak{a}})_0^N(v_\Lambda\otimes 1)=v_\Lambda\otimes(\overline{h}_{\overline{\Lambda}+\rho_\sigma})^{\overline{\tau}}_{(-\frac{1}{2})}\cdot 1-v_\Lambda\otimes(\overline{h}_{(\Lambda+\rho_\sigma)_{|\mathfrak{h}_\mathfrak{a}}})^{\overline{\tau}}_{(-\frac{1}{2})}\cdot 1.$$

Since $h_{\overline{\Lambda}+\rho_\sigma}-h_{(\Lambda+\rho_\sigma)_{|\mathfrak{h}_\mathfrak{a}}}=(h_{\overline{\Lambda}+\rho_\sigma})_\mathfrak{p}$, we obtain

$$(G_{\mathfrak{g},\mathfrak{a}})_0^N(v_\Lambda\otimes 1)=v_\Lambda\otimes(\overline{(h_{\overline{\Lambda}+\rho_\sigma})_\mathfrak{p}})^{\overline{\tau}}_{(-\frac{1}{2})}\cdot 1. \tag{4.20}$$

This proves the first part of the statement.

In particular, by (4.17), $((G_0^N)^2-((G_\mathfrak{a})_0^N)^2)(v_\Lambda\otimes 1)=0$. Now applying Proposition 4.2 to $(G_\mathfrak{a})_0^N$ we see that

$$((G_\mathfrak{a})_0^N)^2(v_\Lambda\otimes 1)=\frac{1}{2}\left((\mu_{|\mathfrak{h}_\mathfrak{a}}+2\rho_{\mathfrak{a}\sigma},\mu_{|\mathfrak{h}_\mathfrak{a}})+\sum_S(\frac{g_S}{12}\dim\mathfrak{a}_S-2g_Sz(\mathfrak{a}_S,\sigma))\right),$$

hence

$$(\mu_{|\mathfrak{h}_\mathfrak{a}}+2\rho_{\mathfrak{a}\sigma},\mu_{|\mathfrak{h}_\mathfrak{a}})+\sum_S(\frac{g_S}{12}\dim\mathfrak{a}_S-2g_Sz(\mathfrak{a}_S,\sigma))\\ =(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda})+\frac{g}{12}\dim\mathfrak{g}-2gz(\mathfrak{g},\sigma).$$

□

**Corollary 4.4.**

$$\|\rho_\sigma\|^2-\|\rho_{\mathfrak{a}\sigma}\|^2=\frac{g}{12}\dim\mathfrak{g}-2gz(\mathfrak{g},\sigma)-\sum_S(\frac{g_S}{12}\dim\mathfrak{a}_S-2g_Sz(\mathfrak{a}_S,\sigma)). \tag{4.21}$$

*Proof.* Plug $\Lambda=-\rho_\sigma$ in (4.19). □

We now observe that $G_{\mathfrak{g},\mathfrak{a}}$ defines a twisted quantum field on $M\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$. Clearly $M\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ is a twisted representation of $V^k(\mathfrak{g})\otimes F(\bar{\mathfrak{p}})$. Recall from Proposition 2.1 that, if we set $\tilde{x}=x-\frac{1}{2}\sum_i:\overline{[x,x_i]}\overline{x}^i:$ for $x\in\mathfrak{g}$, then the map $x\mapsto\tilde{x}$, $\bar{x}\mapsto\bar{x}$ induces an isomorphism $V^k(\mathfrak{g})\otimes F(\bar{\mathfrak{g}})\simeq V^{k+g,1}(R^{super})$. In the next result we show explicitly that $G_{\mathfrak{g},\mathfrak{a}}$ is in the image of $V^k(\mathfrak{g})\otimes F(\bar{\mathfrak{p}})$ under this isomorphism.

**Lemma 4.5.** *Let $\{b_i\}$ be an orthonormal basis of $\mathfrak{p}$. We have*

$$G_{\mathfrak{g},\mathfrak{a}} = \sum_i :\tilde{b}_i\overline{b}_i: -\tfrac{1}{6}\sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{p}}\overline{b}_i\overline{b}_j: . \tag{4.22}$$

*Proof.*

$$\begin{aligned} G_{\mathfrak{g}} - G_{\mathfrak{a}} = \sum_i :b_i\overline{b}_i: &+\tfrac{1}{3}\Big(\sum_{i,j} :\overline{[a_i,b_j]}\overline{a}_i\overline{b}_j: \\ &+\sum_{i,j} :\overline{[b_i,a_j]}\overline{b}_i\overline{a}_j: + \sum_{i,j} :\overline{[b_i,b_j]}\overline{b}_i\overline{b}_j:\Big). \end{aligned} \tag{4.23}$$

First remark that

$$\sum_{i,j} :\overline{[a_i,b_j]}\overline{a}_i\overline{b}_j: = -\sum_{i,j} :\overline{[b_j,a_i]}\overline{a}_i\overline{b}_j: = \sum_{i,j} :\overline{[b_j,a_i]}\overline{b}_j\overline{a}_i:,$$

where the second equality follows from (2.2) since

$$\int_{-T}^{0} [\overline{a}_i\,{}_\lambda\overline{b}_j]d\lambda = T((a_i,b_j)|0\rangle) = 0.$$

Now, since $[a_i,b_j]\in\mathfrak{p}$, using the invariance of the form we get the following relation:

$$\begin{aligned} &\sum_{i,j} :\overline{[a_i,b_j]}\overline{a}_i\overline{b}_j: = \sum_{i,j,k}([a_i,b_j],b_k) :\overline{b}_k\overline{a}_i\overline{b}_j: = \\ &\sum_{i,j,k} :\overline{b}_k([b_j,b_k],a_i)\overline{a}_i\overline{b}_j: = \sum_{j,k} :\overline{b}_k\overline{[b_j,b_k]}_{\mathfrak{a}}\overline{b}_j: . \end{aligned}$$

Finally

$$\sum_{j,k} :\overline{b}_k\overline{[b_j,b_k]}_{\mathfrak{a}}\overline{b}_j: = \sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{a}}\overline{b}_i\overline{b}_j: . \tag{4.24}$$

Indeed by (2.2) and (2.3)

$$\sum_{j,k} :\overline{b}_k:\overline{[b_j,b_k]}_{\mathfrak{a}}\overline{b}_j:: = \sum_{i,j} ::\overline{[b_i,b_j]}_{\mathfrak{a}}\overline{b}_i:\overline{b}_j: = \sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{a}}:\overline{b}_i\overline{b}_j::$$

(here we use several times that $([b_i,b_j]_{\mathfrak{a}},b_k)=0$.) The upshot is that (4.23) simplifies to

$$G_{\mathfrak{g}} - G_{\mathfrak{a}} = \sum_i :b_i\overline{b}_i: + \sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{a}}\overline{b}_i\overline{b}_j: + \frac{1}{3}\sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{p}}\overline{b}_i\overline{b}_j: . \tag{4.25}$$

Now we look at the r.h.s. of (4.22). Using (2.3) and (4.24) we have

$$
\begin{aligned}
\sum_i :\widetilde{b_i}\overline{b}_i: &= \sum_i :b_i\overline{b}_i: -\tfrac{1}{2}\sum_{i,j} ::\overline{[b_i,x_j]}\overline{x}_j:\overline{b}_i: \\
&= \sum_i :b_i\overline{b}_i: -\tfrac{1}{2}\sum_{i,j} ::\overline{[b_i,a_j]}\overline{a}_j:\overline{b}_i: -\tfrac{1}{2}\sum_{i,j} ::\overline{[b_i,b_j]}\overline{b}_j:\overline{b}_i: \\
&= \sum_i :b_i\overline{b}_i: +\tfrac{1}{2}\sum_{i,j} ::\overline{b}_j\overline{[b_i,b_j]}_{\mathfrak{a}}:\overline{b}_i: +\tfrac{1}{2}\sum_{i,j} :\overline{[b_i,b_j]}\overline{b}_i\overline{b}_j: \\
&= \sum_i :b_i\overline{b}_i: +\tfrac{1}{2}\sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{a}}\overline{b}_i\overline{b}_j: +\tfrac{1}{2}\sum_{i,j} :\overline{[b_i,b_j]}\overline{b}_i\overline{b}_j: \\
&= \sum_i :b_i\overline{b}_i: +\sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{a}}\overline{b}_i\overline{b}_j: +\tfrac{1}{2}\sum_{i,j} :\overline{[b_i,b_j]}_{\mathfrak{p}}\overline{b}_i\overline{b}_j: .
\end{aligned}
$$

hence the desired equality (4.22). □

Note that formula (4.22) specializes to (4.5) when $\mathfrak{a} = 0$.

**Remark 4.1.** Set

$$G = \frac{G_{\mathfrak{g},\mathfrak{a}}}{\sqrt{k+g}}, \quad L = \frac{1}{k+g}(\tilde{L}^{\mathfrak{g}} - \tilde{L}^{\mathfrak{a}}) - (L^{\overline{\mathfrak{g}}} - L^{\overline{\mathfrak{a}}}),$$

where $\tilde{L}^{\mathfrak{g}}$ is the image of $L^{\mathfrak{g}} \otimes |0\rangle$ in the isomorphism $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}}) \cong V^{k+g,1}(R^{super})$ of Proposition 2.1, $L^{\mathfrak{g}}$ is defined in (3.7) and $L^{\overline{\mathfrak{g}}}$ is defined in (3.14). A direct computation (cf. [1]) shows that $G$ and $L$ form a Neveu-Schwarz Lie conformal superalgebra

$$NS = \mathbb{C}[T]L + \mathbb{C}[T]G + \mathbb{C}C,$$

$$[L_\lambda L] = (T+2\lambda)L + \frac{\lambda^3}{12}C, \quad [L_\lambda G] = (T + \frac{3}{2}\lambda)G, \quad [G_\lambda G] = 2L + \frac{\lambda^2}{3}C$$

with central charge

$$C = \frac{1}{2}\dim(\mathfrak{p}) - \sum_S (1 - \frac{g_S}{k+g})\dim(\mathfrak{a}_S). \tag{4.26}$$

Set $k = 0$. Then (4.26) vanishes if and only if the pair $(\mathfrak{g},\mathfrak{a})$ is symmetric, i.e. $\mathfrak{a}$ is the algebra of fixed points of an involution of $\mathfrak{g}$. Indeed, if $(\mathfrak{g},\mathfrak{a})$ is symmetric, choosing $\sigma$ as the involutive automorphism that fixes $\mathfrak{a}$, then $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\overline{1/2}}$ and $\mathfrak{a} = \mathfrak{g}^{\bar{0}}$, $\mathfrak{p} = \mathfrak{g}^{\overline{1/2}}$. This implies that $\rho_\sigma = \rho_{\mathfrak{a}\sigma} = \rho_0$ and that $z(\mathfrak{a}_s,\sigma) = 0$ while $z(\mathfrak{g},\sigma) = \frac{1}{16}\dim\mathfrak{p}$. Substituting in (4.21) we find that $C = 0$ in (4.26).

The reverse implication is a consequence of the “Symmetric Space Theorem” by Goddard, Nahm, Olive [2]. We can also derive it from our previous discussion. Indeed choose $\sigma = I$. Then the vanishing of the central charge together with the fact that $F^{\bar\tau}(\bar{\mathfrak{p}})$ is a unitarizable representation of the Ramond Lie superalgebra implies the vanishing of $G$ and $L$. In particular $(G_{\mathfrak{g},\mathfrak{a}})_0^{\mathbb{C}\otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})} = 0$. Writing $G_{\mathfrak{g},\mathfrak{a}}$ explicitly as in Lemma 4.5,

$$0 = (G_{\mathfrak{g},\mathfrak{a}})_0^{\mathbb{C}\otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})} = -\frac{1}{6}\sum_{i,j} :\overline{[b_i,b_j]_{\mathfrak{p}}}\bar b^i\bar b^j :_0^{\mathbb{C}\otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})} .$$

It is easy to check, using Wick's formula, that, if $b, b' \in \mathfrak{p}$, then

$$-\frac{1}{6}\sum_{i,j}[:\overline{[b_i,b_j]_{\mathfrak{p}}}\bar b^i\bar b^j:_\lambda \bar b] = -\frac{1}{2}\sum_i :\overline{[b,b_i]_{\mathfrak{p}}}\bar b^i :$$

and

$$-\frac{1}{2}\sum_i[:\overline{[b,b_i]_{\mathfrak{p}}}\bar b^i :_\lambda \bar b'] = \overline{[b,b']_{\mathfrak{p}}}.$$

This implies that, if we apply $(G_{\mathfrak{g},\mathfrak{a}})_0^{\mathbb{C}\otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})}$ to $\bar b_r^{\mathbb{C}\otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})}\overline{b'_s}^{\mathbb{C}\otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})}\cdot(1\otimes 1)$, then $(\overline{[b,b']_{\mathfrak{p}}})_{r+s}\cdot(1\otimes 1) = 0$ for any $r, s$. This in turns implies that $[b,b']_{\mathfrak{p}} = 0$, hence $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{a}$. Therefore the pair $(\mathfrak{g},\mathfrak{a})$ is symmetric.

Let $\Lambda_0$ be the element of $\widehat{\mathfrak{h}}_0^*$ defined setting $\Lambda_0(d) = \Lambda_0(\mathfrak{h}_0) = 0$ and $\Lambda_0(K) = 1$. Define also $\delta \in \widehat{\mathfrak{h}}_0^*$ setting $\delta(d) = 1$ and $\delta(\mathfrak{h}_0) = \delta(K) = 0$. Set $\widehat{\mathfrak{h}}_{\mathfrak{a}} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathbb{C}d_{\mathfrak{a}} \oplus \sum_S \mathbb{C}K_S$. Let $\delta_{\mathfrak{a}}$ be the analogous element of $\widehat{\mathfrak{h}}_{\mathfrak{a}}^*$ defined by $\delta_{\mathfrak{a}}(K_S) = 0$ for all $S$, $\delta_{\mathfrak{a}}(\mathfrak{h}_{\mathfrak{a}}) = 0$, $\delta_{\mathfrak{a}}(d_{\mathfrak{a}}) = 1$.

Extend $(\cdot,\cdot)$ to all of $\widehat{\mathfrak{h}}_0^*$ by setting $(\Lambda_0,\delta) = 1$ and $(\Lambda_0,\Lambda_0) = (\delta,\delta) = (\delta,\mathfrak{h}_0) = (\Lambda_0,\mathfrak{h}_0) = 0$. Set

$$\widehat{\rho}_\sigma = \rho_\sigma + g\Lambda_0, \quad \widehat{\rho}_{\mathfrak{a}\sigma} = \rho_{\mathfrak{a}\sigma} + \sum_S g_S\Lambda_0^S. \tag{4.27}$$

Then, writing $\Lambda = \overline{\Lambda} + k\Lambda_0 + \Lambda(d)\delta$, we see that

$$(\overline{\Lambda} + 2\rho_\sigma, \overline{\Lambda}) + 2(k+g)\Lambda(d) = \|\Lambda + \widehat{\rho}_\sigma\|^2 - \|\widehat{\rho}_\sigma\|^2. \tag{4.28}$$

Consider the map $\varphi_{\mathfrak{a}} : \mathfrak{h}_0 \oplus (\sum_S \mathbb{C}K_S) \oplus \mathbb{C}d_{\mathfrak{a}} \to \widehat{\mathfrak{h}}_0$

$$\varphi_{\mathfrak{a}}(h) = h \text{ if } h \in \mathfrak{h}_0, \quad \varphi_{\mathfrak{a}}(d_{\mathfrak{a}}) = d, \quad \varphi_{\mathfrak{a}}(K_S) = K \text{ for all } S. \tag{4.29}$$

Since $\varphi_{\mathfrak{a}}$ is onto, $\varphi_{\mathfrak{a}}^*$ is an embedding of $\widehat{\mathfrak{h}}_0^*$ into $(\mathfrak{h}_0 \oplus (\sum_S \mathbb{C}K_S) \oplus \mathbb{C}d_{\mathfrak{a}})^*$. We can therefore view $(\cdot,\cdot)$ as a bilinear form on $\varphi_{\mathfrak{a}}^*(\widehat{\mathfrak{h}}_0^*)$. If $\mu \in \mathfrak{h}_{\mathfrak{a}}^*$ we let $\mu_0$ be

its extension to $\mathfrak{h}_0$ defined by setting $\mu_0(\mathfrak{h}_\mathfrak{p}) = 0$. In this way we can view $\widehat{\mathfrak{h}}_\mathfrak{a}^*$ as a subspace of $(\mathfrak{h}_0 \oplus (\sum_S \mathbb{C}K_S) \oplus \mathbb{C}d_\mathfrak{a})^*$.

In view of Lemma 3.5, we can define the action of $d_\mathfrak{a}$ on $F^{\overline{\tau}}(\overline{\mathfrak{p}})$ by letting it act as

$$(L^{\overline{\mathfrak{p}}})^{\overline{\tau}}_{(1)} - (z(\mathfrak{g},\sigma) - z(\mathfrak{a},\sigma) - \tfrac{1}{16}\dim\mathfrak{p})I_{F^{\overline{\tau}}(\overline{\mathfrak{p}})}. \tag{4.30}$$

With this normalization we have that $d_\mathfrak{a}{\cdot}1 = 0$. The reason for this particular choice will be clear in Proposition 4.6. We can then let $d_\mathfrak{a}$ act on $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ via $d^M \otimes I + I \otimes d^{\overline{\tau}}_\mathfrak{a}$. Given $\nu \in (\widehat{\mathfrak{h}}_\mathfrak{a})^*$, we denote by $(M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}))_\nu$ its $\nu$-weight space. If $(M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}))_\nu \neq 0$, then $\nu + \widehat{\rho}_{\mathfrak{a}\sigma} \in \varphi_\mathfrak{a}^*(\widehat{\mathfrak{h}}_0^*)$ (indeed $\nu + \widehat{\rho}_{\mathfrak{a}\sigma} = \varphi_\mathfrak{a}^*((\nu_{|\mathfrak{h}_\mathfrak{a}})_0 + (\rho_{\mathfrak{a}\sigma})_0 + \nu(d)\delta + (k+g)\Lambda_0))$ and

$$(\nu_{|\mathfrak{h}_\mathfrak{a}} + 2\rho_{\mathfrak{a}\sigma}, \nu_{|\mathfrak{h}_\mathfrak{a}}) + 2(k+g)\nu(d_\mathfrak{a}) = \|\nu + \widehat{\rho}_{\mathfrak{a}\sigma}\|^2 - \|\rho_{\mathfrak{a}\sigma}\|^2. \tag{4.31}$$

If $M$ is a highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$ set $N' = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$. In light of Lemma 4.5, we can consider the operator $(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}$.

**Proposition 4.6.** *If $v \in (M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}))_\nu$, $\mathfrak{n}'_\mathfrak{a} \cdot v = 0$ and $\widehat{\rho}_\sigma$, $\widehat{\rho}_{\mathfrak{a}\sigma}$ are as in* (4.27) *then*

$$((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})^2(v) = \frac{1}{2}(||\Lambda + \widehat{\rho}_\sigma||^2 - ||\nu + \widehat{\rho}_{\mathfrak{a}\sigma}||^2)v.$$

*Proof.* Clearly $(G_{\mathfrak{g},\mathfrak{a}})_0^{N'} \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{a}})} = (G_{\mathfrak{g},\mathfrak{a}})_0^{N}$, so applying (4.18), Lemma 3.2, Lemma 3.3, and using the fact that $L^{\overline{\mathfrak{g}}} - L^{\overline{\mathfrak{a}}} = L^{\overline{\mathfrak{p}}}$, we obtain that

$$\begin{aligned}((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})^2(v) = {} & \left(\frac{1}{2}(\overline{\Lambda} + 2\rho_\sigma, \overline{\Lambda}) + kz(\mathfrak{g},\sigma) + (k+g)\Lambda(d)\right)v \\ & - (k+g)(d^M \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{p}})} + I_M \otimes (L^{\overline{\mathfrak{p}}})^{\overline{\tau}}_{(1)})(v) \\ & - \left(\frac{1}{2}(\nu_{|\mathfrak{h}_\mathfrak{a}} + 2\rho_{\mathfrak{a}\sigma}, \nu_{|\mathfrak{h}_\mathfrak{a}}) + \sum_S (k+g-g_S)z(\mathfrak{a}_S,\sigma)\right)v \\ & - \tfrac{1}{16}\left((k+\frac{g}{3})\dim\mathfrak{g} - \sum_S (k+g-\frac{2g_S}{3})\dim\mathfrak{a}_S\right)v.\end{aligned}$$

By our normalization of the action of $d_\mathfrak{a}$ on $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ we have that

$$(d^M \otimes I_{F^{\overline{\tau}}(\overline{\mathfrak{p}})} + I_M \otimes (L^{\overline{\mathfrak{p}}})^{\overline{\tau}}_{(1)})(v) = (\nu(d_\mathfrak{a}) + z(\mathfrak{g},\sigma) - z(\mathfrak{a},\sigma) - \tfrac{1}{16}\dim\mathfrak{p})v,$$

hence

$$((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})^2(v) = \left(\frac{1}{2}(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda}) + kz(\mathfrak{g},\sigma) + (k+g)\Lambda(d)\right)v$$
$$- (k+g)(\nu(d_{\mathfrak{a}}) + z(\mathfrak{g},\sigma) - z(\mathfrak{a},\sigma) - \tfrac{1}{16}\dim\mathfrak{p})v$$
$$- \left(\frac{1}{2}(\nu_{|\mathfrak{h}_\mathfrak{a}} + 2\rho_{\mathfrak{a}\sigma},\nu_{|\mathfrak{h}_\mathfrak{a}}) + \sum_S (k+g-g_S)z(\mathfrak{a}_S,\sigma)\right)v$$
$$- \tfrac{1}{16}\left((k+\frac{g}{3})\dim\mathfrak{g} - \sum_S (k+g-\frac{2g_S}{3})\dim\mathfrak{a}_S\right)v$$

thus

$$((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})^2(v) = \frac{1}{2}(\|\Lambda+\widehat{\rho}_\sigma\|^2 - \|\nu+\widehat{\rho}_{\mathfrak{a}\sigma}\|^2)v - \frac{1}{2}(\|\rho_\sigma\|^2 - \|\rho_{\mathfrak{a}\sigma}\|^2)v$$
$$+ \frac{1}{2}(\frac{g\dim\mathfrak{g}}{12} - 2gz(\mathfrak{g},\sigma) - \sum_S(\frac{g_S\dim\mathfrak{a}_S}{12} - 2g_Sz(\mathfrak{a}_S,\sigma)))v.$$

Applying Corollary 4.4 we get the result. □

# 5 Multiplets of representations

## 5.1 Kostant's theorem on mutiplets in the twisted affine setting

First of all we study $F^{\overline{\tau}}(\overline{\mathfrak{g}})$ viewed as a $\widehat{L}(\mathfrak{g},\sigma)$-module. The action of $\widehat{L}(\mathfrak{g},\sigma)$ on $F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is obtained by letting $t^j\otimes x$ act via $\theta_{\mathfrak{g}}(x)^{\overline{\tau}}_{(j)}$ where $\theta_{\mathfrak{g}}(x) = x - \tilde{x} = \frac{1}{2}\sum_i :\overline{[x,x_i]}\overline{x}_i:$. In our framework this action corresponds to the pair $(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{g})$ where $\mathfrak{g}$ embeds diagonally in $\mathfrak{g}\oplus\mathfrak{g}$ and the automorphism of $\mathfrak{g}\oplus\mathfrak{g}$ is $\sigma\oplus\sigma$.

Recall from [8, Prop. 6.3] that the choice of a set of positive roots for $\Delta_0$ induces the choice of a set of positive roots $\widehat{\Delta}^+$ for $\widehat{L}(\mathfrak{g},\sigma)$. Let $\widehat{\Pi} = \{\alpha_0,\cdots,\alpha_n\}$ denote the corresponding set of simple roots.

**Lemma 5.1.** *$F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is completely reducible as a $\widehat{\mathfrak{h}}_0$-module and the set of weights of $F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is $\widehat{\rho}_\sigma - S$ where*

$$S = \{\lambda\in\widehat{\mathfrak{h}}_0^* \mid \lambda = \sum_{\alpha\in\widehat{\Delta}^+} n(\alpha)\alpha, \text{ where all but finitely many } n(\alpha) \text{ are zero}$$
$$\text{and each } n(\alpha)\le \operatorname{mult}\alpha\}. \tag{5.1}$$

*Proof.* Fix a basis $\{h_i\}$ of $\mathfrak{h}_0$ such that $(h_i, h_{n-j+1}) = \delta_{ij}$. Choose for any $\alpha \in \mathfrak{h}_0^*$ a basis $\{x_{i\alpha}\}$ of $\mathfrak{g}_\alpha$. If $\gamma \in -\Delta_0^+$, let $y_\gamma$ be a root vector in $\mathfrak{g}_\gamma^{\overline{0}}$. Fix any order in $\Delta_0$ and in the set $\{(i,\alpha) \mid \mathfrak{g}_\alpha \neq 0,\ 1 \leq i \leq \dim \mathfrak{g}_\alpha\}$. Then a basis of $F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is given by the set of vectors

$$(\overline{h}_{i_1})^{\overline{\tau}}_{(-\frac{1}{2})} \cdots (\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})} (\overline{y}_{\gamma_1})^{\overline{\tau}}_{(-\frac{1}{2})} \cdots (\overline{y}_{\gamma_s})^{\overline{\tau}}_{(-\frac{1}{2})} (\overline{x}_{h_1\beta_1})^{\overline{\tau}}_{(j_1)} \cdots (\overline{x}_{h_t\beta_t})^{\overline{\tau}}_{(j_t)}(1) \quad (5.2)$$

where $\left[\frac{n}{2}\right] < i_1 < i_2 < \cdots < i_r \leq n$, $\gamma_1 < \cdots < \gamma_s$, $j_p < -\frac{1}{2}$ for any $p$, and $j_1 \leq \cdots \leq j_t$ with $(h_p, \beta_p) < (h_{p+1}, \beta_{p+1})$ when $j_p = j_{p+1}$. Moreover $\beta_{j_p} \in \Delta_{j_p+\frac{1}{2}}$.

If $h \in \mathfrak{h}_0$, by (9.15), $[(\theta_{\mathfrak{g}}(h))^{\overline{\tau}}_{(0)}, (\overline{a})^{\overline{\tau}}_{(n)}] = (\overline{[h,a]})^{\overline{\tau}}_{(n)}$. By Lemma 3.4 the vector in (5.2) is therefore a weight vector for $\mathfrak{h}_0$ with weight

$$\rho_\sigma + \sum_i \gamma_i + \sum_p \beta_{j_p}.$$

By (4.30) the action of $d$ is given by $(L^{\overline{\mathfrak{g}}})^{\overline{\tau}}_{(1)} - (z(\mathfrak{g},\sigma) - \frac{1}{16}\dim \mathfrak{g}) I_{F^{\overline{\tau}}(\overline{\mathfrak{g}})}$ and, by (3.15), $[(L^{\overline{\mathfrak{g}}})^{\overline{\tau}}_{(1)}, (\overline{x})^{\overline{\tau}}_{(n)}] = (n+\frac{1}{2})(\overline{x})^{\overline{\tau}}_{(n)}$. Since $d \cdot 1 = 0$ we obtain that the vector in (5.2) is an eigenvector for the action of $d$ with eigenvalue $\sum_p (j_p + \frac{1}{2})$.

Finally, by Lemma 3.4, $K$ acts by $2g - g = g$. Summarizing we have that the vector in (5.2) is a weight vector for $\widehat{\mathfrak{h}}_0$ whose weight is

$$\rho_\sigma + g\Lambda_0 + \sum_i \gamma_i + \sum_p ((j_p + \frac{1}{2})\delta + \beta_{j_p}) = \widehat{\rho}_\sigma - \eta,$$

with $\eta = -\sum_i \gamma_i - \sum_p ((j_p + \frac{1}{2})\delta + \beta_{j_p})$. Since $(j_p + \frac{1}{2})\delta + \beta_{j_p}$ can only occur $\dim \mathfrak{g}_{\beta_{j_p}}$ times in the sum, we have that $\eta \in S$. $\square$

**Lemma 5.2.** *Choose a simple root $\alpha_i = s_i\delta + \overline{\alpha}_i$ for $\widehat{L}(\mathfrak{g},\sigma)$ and $x_i \in \mathfrak{g}^{-s_i}_{-\overline{\alpha}_i}$. Then*

$$(G_{\mathfrak{g},\mathfrak{h}_0})^{N'}_0 (\overline{x}_i)^{\overline{\tau}}_{(-s_i-\frac{1}{2})}(1) = 0.$$

*Proof.* We start by computing $[G_{\mathfrak{g},\mathfrak{h}_0}{}_\lambda \overline{x}_i]$. By (4.4) and skewsymmetry of the $\lambda$-bracket, we have $[G_{\mathfrak{g}}{}_\lambda \overline{x}_i] = x_i$. On the other hand, since $\mathfrak{h}_0$ is commutative, $G_{\mathfrak{h}_0} = \sum_j : h_j \overline{h}_j :$, where $\{h_j\}$ is an orthonormal basis of $\mathfrak{h}_0$. It follows from Wick's formula and skewsymmetry that $[G_{\mathfrak{h}_0}{}_\lambda \overline{x}_i] = \sum_j : \overline{[x_i, h_j]} h_j :$. Thus

$$[G_{\mathfrak{g},\mathfrak{h}_0}{}_\lambda \overline{x}_i] = x_i - \sum_j : \overline{[x_i, h_j]} h_j :,$$

or, by writing $x_i = \tilde{x}_i + (x_i - \tilde{x}_i)$,

$$[G_{\mathfrak{g},\mathfrak{h}_0}{}_\lambda \overline{x}_i] = \tilde{x}_i + \frac{1}{2}\sum_t : \overline{[x_i, y_t]}\overline{y}^t : - \sum_j : \overline{[x_i, h_j]}\overline{h}_j :,$$

where, as usual, $\{y_t\}$ and $\{y^t\}$ are a pair of dual basis for $\mathfrak{g}$. Choosing a suitable basis $\{y_i^{\alpha,j}\}$ of $\mathfrak{p} \cap \mathfrak{g}_\alpha^j$, we can assume that $(y_i^{\alpha,r}, y_j^{-\alpha,-r}) = \delta_{ij}$. We choose as basis of $\mathfrak{g}$ the set $(\cup_{\alpha,t,r}\{y_t^{\alpha,r}\}) \cup \{h_j\}$. With this particular choice of basis we can write

$$[G_{\mathfrak{g},\mathfrak{h}_0}{}_\lambda \overline{x}_i] = \tilde{x}_i + \frac{1}{2}\sum_{\alpha,t,r} : \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} : - \frac{1}{2}\sum_j : \overline{[x_i, h_j]}\overline{h}_j : . \tag{5.3}$$

Since $\alpha_i$ is a real root we have that $\dim \mathfrak{g}_{\alpha_i}^{s_i} = 1$, hence

$$[G_{\mathfrak{g},\mathfrak{h}_0}{}_\lambda \overline{x}_i] = \tilde{x}_i + \frac{1}{2}\sum_{(r,\alpha)\neq(s_i,\alpha_i)} : \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} : .$$

We have used the fact that $: \overline{[x_i, y_1^{(\alpha_i,s_i)}]}\overline{y}_1^{-\alpha_i,-s_i} := - : \overline{h}_{\alpha_i}\overline{x}_i := \sum_j : \overline{[x_i, h_j]}\overline{h}_j :$. In particular

$$[(G_{\mathfrak{g},\mathfrak{h}_0})_0^{N'}, (\overline{x}_i)^{\overline{\tau}}_{(-s_i-\frac{1}{2})}] = \frac{1}{2}\sum_{(r,\alpha)\neq(s_i,\alpha_i)} : \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} :^{\overline{\tau}}_{(-s_i)} .$$

Since $\mathfrak{h}_\mathfrak{p} = 0$, by applying (4.20), we have that $(G_{\mathfrak{g},\mathfrak{h}_0})_0^{N'}(1) = 0$, thus we are left with showing that

$$\frac{1}{2}\sum_t \sum_{(r,\alpha)\neq(s_i,\alpha_i)} \left(: \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} :\right)^{\overline{\tau}}_{(-s_i)} (1) = 0.$$

Since $[\overline{[x_i, y_t^{\alpha,r}]}{}_\lambda \overline{y}_t^{-\alpha,-r}] = 0$, we have that

$$Y^{\overline{\tau}}(: \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} :, z) =: Y^{\overline{\tau}}(\overline{[x_i, y_t^{\alpha,r}]}, z)Y^{\overline{\tau}}(\overline{y}_t^{-\alpha,-r}, z) : .$$

By expanding the r.h.s. of the previous equation and picking the coefficient of $z^{s_i-1}$ we find that

$$\begin{aligned}\left(: \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} :\right)^{\overline{\tau}}_{(-s_i)} &= \sum_{n<0} \overline{[x_i, y_t^{\alpha,r}]}^{\overline{\tau}}_{(-s_i+r+\frac{1}{2}+n)}(\overline{y}_t^{-\alpha,-r})^{\overline{\tau}}_{(-n-r-1-\frac{1}{2})} \\ &\quad - \sum_{n\geq 0}(\overline{y}_t^{-\alpha,-r})^{\overline{\tau}}_{(-n-r-1-\frac{1}{2})}\overline{[x_i, y_t^{\alpha,r}]}^{\overline{\tau}}_{(-s_i+r+\frac{1}{2}+n)}.\end{aligned}$$

If $r \neq 0$, we can choose $r \in (-1,0)$ so $(\overline{y}_t^{-\alpha,-r})^{\overline{\tau}}_{(-n-r-1-\frac{1}{2})}(1) = 0$ for $n < 0$, and $\overline{[x_i, y_t^{\alpha,r}]}^{\overline{\tau}}_{(-s_i+r+\frac{1}{2}+n)}(1) = 0$ for $n > 0$. It follows that

$$\left(: \overline{[x_i, y_t^{\alpha,r}]}\overline{y}_t^{-\alpha,-r} :\right)^{\overline{\tau}}_{(-s_i)}(1) = -(\overline{y}_t^{-\alpha,-r})^{\overline{\tau}}_{(-r-1-\frac{1}{2})}\overline{[x_i, y_t^{\alpha,r}]}^{\overline{\tau}}_{(-s_i+r+\frac{1}{2})}(1).$$

Since $(r+1)\delta + \alpha$ is a positive root and $\alpha_i$ is simple, we have that either $[x_i, y_t^{\alpha,r}] = 0$ or $(r+1-s_i)\delta + \alpha - \overline{\alpha}_i$ is still positive. In both cases $\overline{[x_i, y_t^{\alpha,r}]}^{\overline{\tau}}_{(-s_i+r+\frac{1}{2})}(1) = 0$.

It remains to deal with the case $r = 0$, i.e. $\alpha \in \Delta_0$. If $\alpha \in \Delta_0^+$ then either $[x_i, y_t^{\alpha,0}] = 0$ or $s_i = 0$, otherwise $-s_i\delta + \alpha - \overline{\alpha}_i$ would be a positive root. In both cases $\overline{[x_i, y_t^{\alpha,0}]}^{\overline{\tau}}_{(-s_i+\frac{1}{2})}(1) = 0$. If $\alpha \in -\Delta_0^+$ then

$$(\overline{y}_t^{-\alpha,0})^{\overline{\tau}}_{(-\frac{1}{2})}\overline{[x_i, y_t^{\alpha,0}]}^{\overline{\tau}}_{(-s_i-\frac{1}{2})} = -\overline{[x_i, y_t^{\alpha,0}]}^{\overline{\tau}}_{(-s_i-\frac{1}{2})}(\overline{y}_t^{-\alpha,0})^{\overline{\tau}}_{(-\frac{1}{2})}$$

and $(\overline{y}_t^{-\alpha,0})^{\overline{\tau}}_{(-\frac{1}{2})}(1) = 0$. □

**Lemma 5.3.**

$$2\frac{(\widehat{\rho}_\sigma, \alpha_i)}{(\alpha_i, \alpha_i)} = 1.$$

*Proof.* Suppose that $\alpha_i = s_i\delta + \overline{\alpha}_i$. Observe that $(\overline{x}_{-\overline{\alpha}_i})^{\overline{\tau}}_{(-s_i-\frac{1}{2})} \cdot 1 \in F^{\overline{\tau}}(\overline{\mathfrak{p}})$ and that this vector is annihilated by $\mathfrak{n}'_{\mathfrak{h}_0}$. From Lemma 5.2 and Proposition 4.6 (applied with $\mathfrak{a} = \mathfrak{h}_0$, $M = \mathbb{C}$, $\Lambda = 0$) we deduce that $||\widehat{\rho}_\sigma||^2 = ||\nu + \widehat{\rho}_{\mathfrak{h}_0\,\sigma}||^2$, $\nu$ being the weight of $(\overline{x}_{-\overline{\alpha}_i})^{\overline{\tau}}_{(-s_i-\frac{1}{2})} \cdot 1$.

Since $\mathfrak{h}_0$ is commutative, $(\tilde{h})_{\mathfrak{h}_0} = h$ hence $(\tilde{h})_{\mathfrak{h}_0} - \tilde{h} = \theta_{\mathfrak{g}}(h)$. Observe that $(\overline{x}_{-\overline{\alpha}_i})^{\overline{\tau}}_{(-s_i-\frac{1}{2})} \cdot 1$ is a vector of the form given in (5.2). Thus, arguing as in Lemma 5.1, its weight is $\nu = \widehat{\rho}_\sigma - s_i\delta - \overline{\alpha}_i$. We therefore obtain

$$||\rho_\sigma||^2 = ||\rho_\sigma||^2 + ||\alpha_i||^2 - 2(\rho_\sigma\,\overline{\alpha}_i) - 2gs_i$$

or

$$2(\rho_\sigma, \overline{\alpha}_i) = (\alpha_i, \alpha_i) - 2gs_i. \tag{5.4}$$

It follows that

$$2\frac{(\widehat{\rho}_\sigma, \alpha_i)}{(\alpha_i, \alpha_i)} = 1 + \frac{2gs_i}{(\alpha_i, \alpha_i)} - \frac{2gs_i}{(\alpha_i, \alpha_i)}$$

hence the result. □

In the rest of this section we assume that $\mathfrak{a}^{\overline{0}}$ is an equal rank subalgebra of $\mathfrak{g}^{\overline{0}}$. If $\alpha$ is a root of $\widehat{L}(\mathfrak{a},\sigma)$, then $\alpha = k\delta_{\mathfrak{a}} + \overline{\alpha}$. Thus $\alpha = \varphi_{\mathfrak{a}}^*(k\delta + \overline{\alpha})$ ($\varphi_{\mathfrak{a}}$ is as in (4.29)) and a root vector in $\widehat{L}(\mathfrak{a},\sigma)$ for $\alpha$ is a root vector for $k\delta + \overline{\alpha}$ in $\widehat{L}(\mathfrak{g},\sigma)$. It follows that $(\varphi_{\mathfrak{a}}^*)^{-1}$ maps the set of roots $\widehat{\Delta}(\mathfrak{a})$ of $\widehat{L}(\mathfrak{a},\sigma)$ into the set of roots $\widehat{\Delta}$ of $\widehat{L}(\mathfrak{g},\sigma)$. For simplicity we identify $\widehat{\Delta}(\mathfrak{a})$ and $(\varphi_{\mathfrak{a}}^*)^{-1}(\widehat{\Delta}(\mathfrak{a}))$, thus viewing $\widehat{\Delta}(\mathfrak{a})$ as a subset of $\widehat{\Delta}$. Let $\widehat{W}$ be the Weyl group of $\widehat{L}(\mathfrak{g},\sigma)$ and let $\widehat{W}_{\mathfrak{a}}$ be the subgroup generated by the reflections $s_\alpha$ with $\alpha \in \widehat{\Delta}(\mathfrak{a})$. Denote by $\widehat{W}'$ the set of minimal right coset representatives of $\widehat{W}_{\mathfrak{a}}$ in $\widehat{W}$.

If $\Lambda \in \widehat{\mathfrak{h}}_0^*$ is dominant and integral, we let $L(\Lambda)$ denote the irreducible highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$ with highest weight $\Lambda$. We set $\widehat{\Delta}_{\mathfrak{a}}^+ = \widehat{\Delta}^+ \cap \widehat{\Delta}(\mathfrak{a})$. If $\xi \in (\mathfrak{h}_0 \oplus (\sum_S \mathbb{C}K_S) \oplus \mathbb{C}d_{\mathfrak{a}})^*$ is dominant and integral, denote by $V(\xi)$ the irreducible $\widehat{L}(\mathfrak{a},\sigma)$-module with highest weight $\xi$.

The following result is a generalization of Theorem 16 in Landweber's paper [15], where the case $\sigma = I_{\mathfrak{g}}$ is treated.

**Theorem 5.4.** *Assume that $\mathfrak{a}^{\overline{0}}$ is an equal rank subalgebra of $\mathfrak{g}^{\overline{0}}$ and that $\Lambda$ is a dominant integral weight for $\widehat{L}(\mathfrak{g},\sigma)$. Set $X = L(\Lambda) \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$. Then*

$$Ker\,(G_{\mathfrak{g},\mathfrak{a}})_0^X = \bigoplus_{w \in \widehat{W}'} V(\varphi_{\mathfrak{a}}^*(w(\Lambda + \widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma}). \tag{5.5}$$

*Proof.* Suppose that $V(\xi)$ occurs in $Ker\,((G_{\mathfrak{g},\mathfrak{a}})_0^X)^2$. Then $\xi = \gamma + \beta$ where $\gamma$ is a weight of $F^{\overline{\tau}}(\overline{\mathfrak{p}})$ and $\beta$ is a weight of $L(\Lambda)$. By Lemma 5.1, we know the form of the weights of $F^{\overline{\tau}}(\overline{\mathfrak{g}})$. Since we are assuming that $\mathrm{rank}(\mathfrak{a}^{\overline{0}}) = \mathrm{rank}(\mathfrak{g}^{\overline{0}})$ the weights of $F^{\overline{\tau}}(\overline{\mathfrak{p}}) \otimes 1 \subset F^{\overline{\tau}}(\mathfrak{g})$ are also of this form. Hence we can write $\xi + \widehat{\rho}_{\mathfrak{a}\sigma} = \varphi_{\mathfrak{a}}^*(-\nu + \beta + \widehat{\rho}_\sigma)$ where $\nu \in S$ (cf. (5.1)). By Proposition 4.6 we have that $||-\nu + \beta + \widehat{\rho}_\sigma||^2 = ||\Lambda + \widehat{\rho}_\sigma||^2$. Lemma 5.3 tells us that $\widehat{\rho}_\sigma$ is what is usually denoted by $\rho$ for $\widehat{L}(\mathfrak{g},\sigma)$. Hence we can use [14, Lemma 3.2.4] to deduce the existence of $w \in \widehat{W}$ such that $\xi = \varphi_{\mathfrak{a}}^*(w(\Lambda + \widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma}$. We claim that $w \in \widehat{W}'$ and that for any $w \in \widehat{W}'$ the corresponding submodule occurs with multiplicity one in $Ker\,((G_{\mathfrak{g},\mathfrak{a}})_0^X)^2$. The proof of all these statements can be done along the lines of Kostant's argument in the finite dimensional case, as extended to the affine case by Kumar in [14, Theorem 3.2.7]. There is only one difference with Kumar's setting: $\widehat{W}_{\mathfrak{a}}$ is a reflection subgroup of $\widehat{W}$ and not, in general, a standard parabolic subgroup. But Kumar's proof relies on a description of $\widehat{W}'$ (see [14, Exercise 1.3.E]) which holds in our weaker hypothesis too. This concludes the proof since, by Proposition 9.2 below, $(G_{\mathfrak{g},\mathfrak{a}})_0^X$ is self-adjoint (in our hypothesis), hence $Ker\,(G_{\mathfrak{g},\mathfrak{a}})_0^X = Ker\,((G_{\mathfrak{g},\mathfrak{a}})_0^X)^2$. □

## 5.2 Applications

We want to discuss in our setting some consequences of Theorem 5.4 which are the analogues of Theorems 4.17 and 4.24 of [12] in the finite dimensional case. As in the finite dimensional case, we name *multiplet* the set of $\widehat{L}(\mathfrak{a},\sigma)$-modules occurring in the decomposition (5.5). As discussed in the Introduction, these were discovered in [3] (in the finite dimensional equal rank case), where it is shown that they possess remarkable properties. First, the Casimir element acts by the same scalar on all the representations in the multiplets. This fact has a direct analogue in the affine case. Indeed $C(\mathfrak{a}) = (L_0^{\mathfrak{a}})^X + (k+g)d_{\mathfrak{a}}^X$ can be considered as (one half of the) Casimir element for $\widehat{L}(\mathfrak{a},\sigma)$ acting on $X = L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})$: this follows e.g. from the formula displayed in [8, Exercise 7.16], noting that in our context the central elements $K_S$ specialize to the levels $k+g-g_S$. We shall deduce the above remarkable property by a formula for the square of the Dirac operator acting on $L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})$, which holds in the framework of Section 4 (i.e., $(\mathfrak{g},\mathfrak{a})$ a reductive pair, $\Lambda$ any weight of $\widehat{L}(\mathfrak{g},\sigma)$). The following result is a twisted affine analog of [12, Theorem 2.13].

**Proposition 5.5.** *Set $N = M\otimes F$, where $M$ is any level $k$ highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$ and $F = F^{\overline{\tau}}(\overline{\mathfrak{p}})$. Set $C(\mathfrak{g}) = (L_0^{\mathfrak{g}})^M + (k+g)d^M$, $C(\mathfrak{a}) = (L_0^{\mathfrak{a}})^N + (k+g)d_{\mathfrak{a}}^N$. Then*

$$((G_{\mathfrak{g},\mathfrak{a}})_0^N)^2 = C(\mathfrak{g})\otimes I_F - C(\mathfrak{a}) + \left[\frac{1}{2}(\|\rho_\sigma\|^2 - \|\rho_{\mathfrak{a}\sigma}\|^2) + c(k)\right] I_N, \tag{5.6}$$

*where $c(k) = -kz(\mathfrak{g},\sigma) + \sum_S (k+g-g_S)z(\mathfrak{a}_S,\sigma)$.*

*Proof.* Combine formulas (4.18), (4.30), and (4.21). □

**Corollary 5.6.** *Under the hypothesis of Theorem 5.4, $C(\mathfrak{a})$ acts on all $\widehat{L}(\mathfrak{a},\sigma)$-modules $V(\varphi_{\mathfrak{a}}^*(w(\Lambda+\widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma})$ of the multiplet* (5.5) *by the scalar*

$$\frac{1}{2}\left(\|\Lambda+\widehat{\rho}_\sigma\|^2 - \|\rho_{\mathfrak{a}\sigma}\|^2\right) + \sum_S (k+g-g_S)z(\mathfrak{a}_S,\sigma). \tag{5.7}$$

*Proof.* By Theorem 5.4 and formula (5.6), $C(\mathfrak{a})$ acts as

$$C(\mathfrak{g})\otimes I_F + \left[\frac{1}{2}(\|\rho_\sigma\|^2 - \|\rho_{\mathfrak{a}\sigma}\|^2) + c(k)\right] I_N$$

on the multiplet. Combine Lemma 3.3 and formula (4.28) to compute the action of $C(\mathfrak{g})$. □

The second property discovered in [3] involves the dimensions of the representations in the multiplets. In the finite dimensional case the multiplets are canonically indexed by the set $W'$ of minimal length representatives of (right) cosets of the Weyl group of $\mathfrak{a}$ (a reductive subalgebra of $\mathfrak{g}$ of the same rank) in the Weyl group $W$ of $\mathfrak{g}$. If we let $V_w$ denote the $\mathfrak{g}$-module indexed by $w \in W'$, then

$$\sum_{w \in W'} (-1)^{\ell(w)} \dim V_w = 0, \tag{5.8}$$

where $\ell(\cdot)$ is the length function on $W$.

Obviously this result cannot hold true in the affine case for the representations involved are infinite dimensional. However in Proposition 5.7 below we obtain an analog of (5.8) involving the asymptotic dimensions of the representations $V(\varphi_{\mathfrak{a}}^*(w(\Lambda + \widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma})$ as defined in [8, Ch. 13].

To prove this fact we need several preliminary considerations. Let $V$ be a complex vector space endowed with a symmetric bilinear form $(\ ,\ )$. Fix $\sigma \in O(V)$ and assume that $\sigma$ is diagonalizable with modulus 1 eigenvalues. Suppose also that the set of $\sigma$-fixed points is even dimensional. Set $\tilde{V} = V \oplus \mathbb{C}$ and extend $(\ ,\ )$ to $\tilde{V}$ by setting $(v, 1) = 0$, $(1, 1) = 1$. Then $so(V)$ embeds in $so(\tilde{V})$ and $(so(\tilde{V}), so(V))$ is a reductive pair. Indeed, for $v \in V$ define $X_v \in so(\tilde{V})$ by $X_v(w+c) = cv - (v, w)$. If we endow $so(\tilde{V})$ with the invariant form $\langle X, Y\rangle = \frac{1}{2}tr(XY)$, then we have that $so(V)^\perp = \{X_v \mid v \in V\}$. Note that, if $A \in so(V)$ and $v \in V$, then $[A, X_v] = X_{A(v)}$. Thus, identifying $V$ with $\{X_v \mid v \in V\}$, we see that the adjoint action of $so(V)$ on its orthogonal complement gets identified with the natural action of $so(V)$ on $V$.

Extend $\sigma$ to an automorphism of $\tilde{V}$ by letting $\sigma(1) = 1$. Then $\sigma X_v \sigma^{-1} = X_{\sigma(v)}$. Let $\overline{V}$ be the space $V$ viewed as an odd space, and set $\overline{\tau} = -\sigma$. By applying our machinery to the reductive pair $(so(\tilde{V}), so(V))$ we can turn $F^{\overline{\tau}}(\overline{V})$ into a $\widehat{L}(so(V), Ad(\sigma))$-module. Let $\sigma_0 \in End(\tilde{V})$ be defined by $\sigma_0(v) = v$, $\sigma_0(1) = -1$. Then the decomposition $so(\tilde{V}) = so(V) \oplus V$ is precisely the eigenspace decomposition of $\sigma_0$. Since $\sigma_0^2 = I$, the pair $(so(\tilde{V}), so(V))$ is actually symmetric. As observed in Remark 4.1, we have that $(G_{so(\tilde{V}),so(V)})_0$ acts trivially on $L(\overline{V}, \overline{\tau})$. Thus $L(\overline{V}, \overline{\tau})$ decomposes as a $\widehat{L}(so(V), Ad(\sigma))$-module as prescribed by Theorem 5.4. Hence we need to find the set of minimal right coset representatives of the Weyl group of $\widehat{L}(so(V), Ad(\sigma))$ in the Weyl group of $\widehat{L}(so(\tilde{V}), Ad(\sigma))$-module. For this we need to distinguish two cases. Suppose first that $\det \sigma = 1$. It follows that $Ad(\sigma)$ is an automorphism of $so(V)$ of inner type. Choose a Cartan subalgebra $\mathfrak{h}_{so(V)}$ of $so(V)$ which is fixed by $Ad(\sigma)$. Since $\det \sigma = 1$, $\dim(V)$ is even (recall that we are assuming that the set of $\sigma$-fixed vectors in $V$ is even dimensional), hence $\mathfrak{h}_{so(V)}$ is a Cartan subalgebra of $so(\tilde{V})$. Choose

$h \in \mathfrak{h}_{so(V)}$ such that $Ad(\sigma) = e^{2\pi i\, ad(h)}$. Let $\{\beta_1, \dots, \beta_n\}$ be a set of simple roots for $so(\tilde V)$. Let $\theta$ be the corresponding highest root for $so(\tilde V)$. Then we can assume that $\beta_i(h) \geq 0$ $i = 1, \dots, n$, $\theta(h) \leq 1$. It follows that the map $n\delta + \alpha \mapsto (n + \alpha(h))\delta + \alpha$ is a bijection between the set of real roots of $\widehat{L}(so(\tilde V), I)$ and the set of real roots of $\widehat{L}(so(\tilde V), Ad(\sigma))$ that maps $\{\delta-\theta, \beta_1, \dots, \beta_n\}$ to the set $\{\alpha_0, \dots, \alpha_n\}$ of simple roots for $\widehat{L}(so(\tilde V), Ad(\sigma))$ and the set $\{\delta - \theta, \beta_1, \dots, \beta_{n-1}, s_{\beta_n}(\beta_{n-1})\}$ of simple roots of $\widehat{L}(so(V), I)$ to the set of simple roots of $\widehat{L}(so(V), Ad(\sigma))$. Then the Weyl groups of $\widehat{L}(so(\tilde V), Ad(\sigma))$ and $\widehat{L}(so(V), Ad(\sigma))$ are isomorphic to the Weyl groups of $\widehat{L}(so(\tilde V), I)$ and $\widehat{L}(so(V), I)$, respectively, the index of the latter in the former is two, and the set of minimal length coset representatives is $\{1, s_{\alpha_n}\}$. Note finally that, in this case, $\varphi_{so(V)}$ is the identity. Thus, according to Theorem 5.4,

$$F^{\bar\tau}(\overline{V}) = V(\widehat{\rho}_{Ad(\sigma)} - \widehat{\rho}_{so(V),Ad(\sigma)}) \oplus V(s_{\alpha_n}(\widehat{\rho}_{Ad(\sigma)} - \widehat{\rho}_{so(V),Ad(\sigma)})). \tag{5.9}$$

(see (3.6), (3.22) for notation). Here we used the fact that $s_{\alpha_n}(\widehat{\rho}_{so(V),Ad(\sigma)}) = \widehat{\rho}_{so(V),Ad(\sigma)}$.

If instead $\det\sigma = -1$ then $\dim V$ is odd and $Ad(\sigma)$ is of inner type for $so(V)$ but not for $so(\tilde V)$. Let, as above, $\mathfrak{h}_{so(V)}$ be a Cartan subalgebra of $so(V)$ fixed pointwise by $\sigma$. Then there is an element $h$ of $\mathfrak{h}_{so(V)}$ such that $Ad(\sigma) = Ad(\sigma_0)e^{2\pi i ad(h)}$. Arguing as in the $\det\sigma = 1$ case, we find that the index of the Weyl group of $\widehat{L}(so(V), Ad(\sigma))$ in the Weyl group of $\widehat{L}(so(\tilde V), Ad(\sigma))$ equals the index of the Weyl group of $\widehat{L}(so(V), I)$ in the Weyl group of $\widehat{L}(so(\tilde V), Ad(\sigma_0))$ and that the set of minimal length representatives is $\{I, s_{\alpha_0}\}$. Hence, in this case

$$F^{\bar\tau}(\overline{V}) = V(\widehat{\rho}_{Ad(\sigma)} - \widehat{\rho}_{so(V),Ad(\sigma)}) \oplus V(s_{\alpha_0}(\widehat{\rho}_{Ad(\sigma)} - \widehat{\rho}_{so(V),Ad(\sigma)})). \tag{5.10}$$

On the algebra $Cl(L(\overline{V}, \bar\tau))$ there is a unique involutive automorphism such that $x \mapsto -x$ for $x \in L(\overline{V}, \sigma)$. Then, denoting by $Cl(L(\overline{V}, \bar\tau))^{\pm}$ the $\pm 1$ eigenspace for this automorphism, we can write

$$Cl(L(\overline{V}, \bar\tau)) = Cl(L(\overline{V}, \bar\tau))^{+} \oplus Cl(L(\overline{V}, \bar\tau))^{-}.$$

Recall from § 3.2 that

$$F^{\bar\tau}(\overline{V}) = Cl(L(\overline{V}, \bar\tau))/Cl(L(\overline{V}, \bar\tau))L^{+}(\overline{V}, \bar\tau).$$

It follows that

$$F^{\bar\tau}(\overline{V}) = F^{\bar\tau}(\overline{V})^{+} \oplus F^{\bar\tau}(\overline{V})^{-},$$

where $F^{\bar\tau}(\overline{V})^{\pm} = Cl(L(\overline{V},\bar\tau))^{\pm}/(Cl(L(\overline{V},\bar\tau))L^{+}(\overline{V},\bar\tau)\cap Cl(L(\overline{V},\bar\tau))^{\pm}$. Moreover $Cl(L(\overline{V},\bar\tau))^{+}$ acts naturally on $F^{\bar\tau}(\overline{V})^{\pm}$.

Set $\theta_{so(V)}(X) = \frac{1}{2}\sum : \overline{X(b_i)}\bar{b}^i :$, where $\{b_i\}$, $\{b^i\}$ are bases of $V$ dual to each other. We note that the action of $X_r \in \widehat{L}(so(V), Ad(\sigma))$ is given by $(\theta_{so(V)}(X))_r^{F^{\bar\tau}(\overline{V})}$. It follows that $F^{\bar\tau}(\overline{V})^{\pm}$ are stable under the action of $\widehat{L}(so(V), Ad(\sigma))$. Thus, by the decompositions (5.9), (5.10), we obtain that $F^{\bar\tau}(\overline{V})^{\pm}$ are both irreducible $\widehat{L}(so(V), Ad(\sigma))$-modules whose highest weights are switched by an involution $s$ of the Dynkin diagram of $\widehat{L}(so(V), Ad(\sigma))$.

In particular, if $(\mathfrak{g},\mathfrak{a})$ is any reductive pair and $\sigma$ is an automorphism such that rank $\mathfrak{a}^{\bar 0}$ = rank $\mathfrak{g}^{\bar 0}$, we can apply the above discussion to $F^{\bar\tau}(\bar{\mathfrak{p}})$, turning it into a $\widehat{L}(so(\mathfrak{p}), Ad(\sigma))$-module. Note that we can see $L'(\mathfrak{a},\sigma)$ as a subalgebra of $L'(so(\mathfrak{p}), Ad(\sigma))$ by embedding $\mathfrak{a}$ in $so(\mathfrak{p})$ via $ad_{\mathfrak{p}}$ and that the action of $L'(\mathfrak{a},\sigma)$ on $F^{\bar\tau}(\bar{\mathfrak{p}})$ is just the restriction of the action of $L'(so(\mathfrak{p}), Ad(\sigma))$. Since, by Wick's formula,

$$[\theta_{so(\mathfrak{p})}(X)_\lambda L^{\overline{\mathfrak{p}}}] = -\lambda\theta_{so(V)}(X),$$

letting $d_{so(V)}$ act as $(L^{\overline{\mathfrak{p}}})_0^{\bar\tau} - (z(\mathfrak{g},\sigma) - z(\mathfrak{a},\sigma) - \frac{1}{16}\dim\mathfrak{p})I$, we can extend this action to $\widehat{L}(so(\mathfrak{p}), Ad(\sigma))$ in such a way that the action of $d_{so(\mathfrak{p})}$ equals the action of $d_{\mathfrak{a}}$.

Now we observe that in our setting the so-called "homogeneous Weyl-Kac" formula holds (recall that $\mathfrak{a}^{\overline{0}}$ is assumed to have the same rank of $\mathfrak{g}$). Indeed, in the (completed) representation ring of $\widehat{L}(\mathfrak{a},\sigma)$, we have

$$L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})^{+} - L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})^{-} = \sum_{w\in\widehat{W}'}(-1)^{\ell(w)}V(\varphi_{\mathfrak{a}}^{*}(w(\Lambda+\widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma}). \tag{5.11}$$

This relation can be proved exactly as in the finite dimensional case (or affine $\sigma = I$ case, see [15, Theorem 4]), using Lemma 5.1 to evaluate $F^{\overline{\tau}}(\overline{\mathfrak{p}})^{+} - F^{\overline{\tau}}(\overline{\mathfrak{p}})^{-}$.

**Remark 5.1.** If $\mu \in \widehat{\mathfrak{h}}_{\mathfrak{a}}^{*}$ is dominant integral for $\widehat{\Delta}_{\mathfrak{a}}^{+}$ and $w \in \widehat{W}_{\mathfrak{a}}$, we set $V(w(\mu+\widehat{\rho}_{\mathfrak{a},\sigma}) - \widehat{\rho}_{\mathfrak{a},\sigma}) = (-1)^{\ell(w)}V(\mu)$. Then, with this definition, we can rewrite (5.11) as

$$L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})^{+} - L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})^{-} = \sum_{x\in\widehat{W}_{\mathfrak{a}}\backslash\widehat{W}}(-1)^{\ell(w_x)}V(\varphi_{\mathfrak{a}}^{*}(w_x(\Lambda+\widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma}), \tag{5.12}$$

where $\widehat{W}_{\mathfrak{a}}\backslash\widehat{W}$ is any set of right coset representatives and $w_x$ is any element from the coset $x$.

Following [8, Ch. 13] or [6, (2.2.1)], we recall the asymptotics of the character of an integrable highest weight module $V$ over the affine algebra $\widehat{L}(\mathfrak{a},\sigma)$, where $\mathfrak{a}$ is a simple or abelian Lie algebra. Recall that the series

$$ch_V(\tau,h) = tr_V e^{2\pi i(-\tau d_{\mathfrak{a}}+h)}$$

converges to an analytic function of the complex variable $\tau$, if $Im\,\tau > 0$, for each $h \in \mathfrak{h}_{\mathfrak{a}}$. The asymptotics of this function, as $\tau \downarrow 0$ (i.e. $\tau = it$, $t \in \mathbb{R}^+, t \to 0$), is as follows:

$$ch_V(\tau,h) \approx a(\Lambda)e^{\frac{\pi i c(k)}{12\tau}}, \tag{5.13}$$

where $V = V(\Lambda)$ is a highest weight module with highest weight $\Lambda$, $c(k)$ is the conformal anomaly (= Sugawara central charge, see [6, (1.4.2)]), which depends only on the level $k = \Lambda(K)$ of $V$, and

$$a(\Lambda) = b(k)\prod_{\alpha\in R^+}\sin\pi\frac{(\bar{\Lambda}+\rho_{\mathfrak{a}},\alpha)}{k+g_{\mathfrak{a}}}. \tag{5.14}$$

Here $b(k)$ is a positive constant, depending only on $k$ (one can find in [8] a simple formula for $b(k)$, which is unimportant for the present paper) and $R^+$ denotes the set of positive roots (resp. coroots) of $\mathfrak{a}$ if $\widehat{L}(\mathfrak{a},\sigma)$ is of type $X_n^{(1)}$ or $A_{2n}^{(2)}$ (resp. all other types). Moreover, $\rho_{\mathfrak{a}} = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ and $g_{\mathfrak{a}}$ is half of the value of the Casimir operator on $\mathfrak{a}$.

Now let $\mathfrak{a}$ be a reductive Lie algebra and let $\mathfrak{a} = \bigoplus_{j=0}^{s}\mathfrak{a}_j$ be the decomposition of $\mathfrak{a}$ in the direct sum of an abelian Lie algebra $\mathfrak{a}_0$ and simple components $\mathfrak{a}_j$, $j \geq 1$. Let $V$ be an integrable $\widehat{L}(\mathfrak{a},\sigma)$-module form the category $\mathcal{O}$ of level $k = (k_0,\dots,k_s)$ (i.e., $K_j \in \widehat{L}(\mathfrak{a}_j,\sigma)$ acts on $V$ via the scalar $k_j$). Motivated by the above discussion, we define the *asymptotic dimension* of $V$ by

$$\operatorname{asdim}(V) = \lim_{\tau\downarrow 0} e^{-\frac{\pi i}{12\tau}\sum_{j=0}^{s}c(k_j)}ch_V(\tau,h). \tag{5.15}$$

If $V$ is irreducible, then it is an outer tensor product of irreducible $\widehat{L}(\mathfrak{a}_j,\sigma)$-modules with highest weights $\Lambda^j$ of level $k_j$, $j = 0,\dots,s$, and it follows from (5.13) that

$$\operatorname{asdim}(V) = \prod_{j=1}^{s}a(\Lambda^j),$$

where $a(\Lambda)$ is given by (5.14). We stipulate that $\operatorname{asdim}(V) = 1$ if $s = 0$, i.e. if $\mathfrak{a}$ is abelian.

**Remark 5.2.** Note that the asymptotic dimension is a positive real number, which has all properties of the usual dimension. Namely, the asymptotic dimension of the tensor product of modules equals the product of asymptotic dimensions of the factors, and the asymptotic dimension of a finite direct sum of modules of the same level equals the sum of asymptotic dimensions of summands. In particular we can extend the asymptotic dimension by linearity to the subring of the representation ring of $\widehat{L}(\mathfrak{a}, \sigma)$ generated by the integrable highest weight modules.

Note that $\widehat{W}_{\mathfrak{a}}$ has finite index in $\widehat{W}$ iff $\mathfrak{a}^{\bar{0}}$ is a semisimple equal rank Lie subalgebra of $\mathfrak{g}^{\bar{0}}$. To deal with the reductive case we need some preliminaries and more notation. Let $M \subset \mathfrak{h}_0$ be the lattice which indexes the translations in the Weyl group of $\widehat{L}(\mathfrak{g}, \sigma)$ (see [8, (6.5.8)]) and set $M_0 = \mathfrak{a}_0^{\bar{0}} \cap M$ (recall that $\mathfrak{a}_0^{\bar{0}}$ is the center of $\mathfrak{a}^{\bar{0}}$). Let $P_0$ be the lattice in $\mathfrak{a}_0^{\bar{0}}$ dual to $M_0$ and let $L_0$ the lattice corresponding to $M_0$ in $\mathfrak{h}_0^*$ under the identification induced by the bilinear form. Let $T_{L_0} = \{t_\alpha \mid \alpha \in L_0\}$ be the set of translations by elements of $L_0$ (see [8, (6.5.2)]). Let $\widehat{W}'_{fin}$ be a set of representatives for the right cosets of $T_{L_0} \times \widehat{W}_{\mathfrak{a}}$ in $\widehat{W}$. Note that in the semisimple case $\widehat{W}'_{fin}$ is a set of right coset representatives for $\widehat{W}_{\mathfrak{a}}$ in $\widehat{W}$.

Set

$$\widehat{\mathfrak{h}}_c^* = (\mathfrak{a}_0^{\bar{0}})^* \oplus \mathbb{C}\Lambda_0^0, \quad \widehat{\mathfrak{h}}_{ss}^* = (\mathfrak{h}_0 \cap \sum_{S>0} \mathfrak{a}_S)^* \oplus \sum_{S>0} \mathbb{C}\Lambda_0^S,$$

so that any $\lambda \in \widehat{\mathfrak{h}}_{\mathfrak{a}}^*$ can be uniquely written as $\lambda = \lambda_c + \lambda_{ss} + a\delta_{\mathfrak{a}}$, $\lambda_c \in \widehat{\mathfrak{h}}_c^*, \lambda_{ss} \in \widehat{\mathfrak{h}}_{ss}^*, a \in \mathbb{C}$. Set $r = \dim \mathfrak{a}_0^{\bar{0}}$.

**Proposition 5.7.** *If $\mathfrak{a}^{\bar{0}}$ is a reductive equal rank subalgebra of $\mathfrak{g}^{\bar{0}}$ such that $\mathfrak{a}_0^{\bar{0}} = Span_{\mathbb{C}} M_0$, then $\widehat{W}'_{fin}$ is finite and*

$$\sum_{w \in \widehat{W}'_{fin}} (-1)^{\ell(w)} \mathrm{asdim}(V(\varphi_{\mathfrak{a}}^*(w(\Lambda + \widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma})) = 0.$$

*(here* $\mathrm{asdim}(V(\varphi_{\mathfrak{a}}^*(w(\Lambda + \widehat{\rho}_\sigma)) - \widehat{\rho}_{\mathfrak{a}\sigma}))$ *is defined as in Remark 5.2).*

*Proof.* Let $M_{\mathfrak{a}}$ be the lattice which indexes the translations in the Weyl group of $\widehat{L}(\mathfrak{a}, \sigma)$. Since $\mathfrak{h}_0 \subset \mathfrak{a}$, $\mathrm{rank}(M_0 \oplus M_{\mathfrak{a}}) = \mathrm{rank}\, \mathfrak{a} = \dim \mathfrak{h}_0 = \mathrm{rank}\, M$, hence $T_{L_0} \times \widehat{W}_{\mathfrak{a}}$ has finite index in $\widehat{W}$. This proves the first claim.

Let $\Lambda^\pm$ be the highest weights of $F^{\bar{\tau}}(\bar{\mathfrak{p}})^\pm$ and let $s$ be the involution of the Dynkin diagram of $\widehat{L}(so(\mathfrak{p}), Ad(\sigma))$ such that $s\Lambda^+ = \Lambda^-$. Let $h$ be an element in a Cartan subalgebra $\mathfrak{h}_{so(\mathfrak{p})}$ of $so(\mathfrak{p})$ such that $d_{so(\mathfrak{p})} + h$ is the unique element of $\mathbb{C}d + \mathfrak{h}_{so(\mathfrak{p})}$ such that $\gamma(d_{so(\mathfrak{p})} + h) = 1$ for any simple root

$\gamma$ of $\widehat{L}(so(\mathfrak{p}), Ad(\sigma))$. Let $ch^{\pm}$ be the character of $F^{\bar{\tau}}(\bar{\mathfrak{p}})^{\pm}$ as $\widehat{L}(so(\mathfrak{p}), Ad(\sigma))$-modules. Since $s(d_{so(\mathfrak{p})} + h) = d_{so(\mathfrak{p})} + h$ we see that

$$(ch^+ - ch^-)(\tau, h) = 0.$$

In particular we see that

$$\lim_{\tau \downarrow 0} (ch^+ - ch^-)(\tau, h) = 0.$$

This limit is independent of $h$, thus $(ch^+ - ch^-)(\tau, 0) \approx 0$ as $\tau \downarrow 0$. Since $d_{so(\mathfrak{p})}$ acts as $d_{\mathfrak{a}}$ we get for the $\widehat{L}(\mathfrak{a}, \sigma)$-modules:

$$(ch_{L(\Lambda)\otimes F^{\bar{\tau}}(\bar{\mathfrak{p}})^+} - ch_{L(\Lambda)\otimes F^{\bar{\tau}}(\bar{\mathfrak{p}})^-})(\tau, 0) \approx 0, \tag{5.16}$$

as $\tau \downarrow 0$.

Define $m_w = w(\Lambda + \widehat{\rho}_\sigma)(d)$ and $\Lambda^w = w(\Lambda + \widehat{\rho}_\sigma)$. Observe that, for $\alpha \in L_0$ there exists constants $c_1, c_2$ such that $t_\alpha(\lambda) = \lambda + c_1\lambda(K)\alpha - c_2((\lambda, K) + \frac{1}{2}|\alpha|^2)\delta$. Hence we have

$$\begin{aligned}\varphi_{\mathfrak{a}}^*(t_\alpha\Lambda^w) = \varphi_{\mathfrak{a}}^*(\Lambda^w)_c + c_1(k+g)\alpha - c_2((\varphi_{\mathfrak{a}}^*(\Lambda^w)_c, \alpha) + \frac{1}{2}|\alpha|^2)\delta_{\mathfrak{a}} \\ + \varphi_{\mathfrak{a}}^*(\Lambda^w)_{ss} + m_w\delta_{\mathfrak{a}}.\end{aligned}$$

Setting, for $\mu \in \widehat{\mathfrak{h}}_c^*$, $\dot{t}_\alpha(\mu) = \mu + c_1(k+g)\alpha - c_2((\mu, \alpha) + \frac{1}{2}|\alpha|^2)\delta_{\mathfrak{a}}$, we can write

$$ch(V(\varphi_{\mathfrak{a}}^*(t_\alpha\Lambda^w) - \widehat{\rho}_{\mathfrak{a}\sigma}) = \frac{e^{\dot{t}_\alpha(\varphi_{\mathfrak{a}}^*(\Lambda^w)_c))}}{\eta(\tau)^r} chV(\varphi_{\mathfrak{a}}^*(\Lambda^w)_{ss} + m_w\delta_{\mathfrak{a}} - \widehat{\rho}_{\mathfrak{a}\sigma}),$$

$\eta$ being the Dedekind $\eta$-function.

Since $T_{L_0}\widehat{W}'_{fin}$ is a set of coset representatives for $\widehat{W}_{\mathfrak{a}}$ in $\widehat{W}$ and observing that multiplying any element $w \in \widehat{W}$ by a translation does not change the parity of $\ell(w)$, we can rewrite (5.12) as

$$\begin{aligned}&ch(L(\Lambda) \otimes F^{\bar{\tau}}(\bar{\mathfrak{p}})^+) - ch(L(\Lambda) \otimes F^{\bar{\tau}}(\bar{\mathfrak{p}})^-) = \\ &\sum_{w\in\widehat{W}'_{fin}} \sum_{\alpha\in L_0} (-1)^{\ell(t_\alpha w)} ch(V(\varphi_{\mathfrak{a}}^*(t_\alpha\Lambda^w) - \widehat{\rho}_{\mathfrak{a}\sigma})) = \\ &\sum_{w\in\widehat{W}'_{fin}} (-1)^{\ell(w)} \frac{\Theta(\Lambda_c^w)}{\eta(\tau)^r} ch(V(\varphi_{\mathfrak{a}}^*(\Lambda^w)_{ss} - \widehat{\rho}_{\mathfrak{a}\sigma} + m_w\delta_{\mathfrak{a}}))\end{aligned}$$

where $\Theta(\mu) = \sum_{\alpha \in L_0} e^{i_\alpha(\mu)}$. Hence, by the asympotics given in [11, (2.4.3)] and (5.16), we find that, taking the limit as $\tau \downarrow 0$,

$$0 = \sum_{w \in \widehat{W}'_{fin}} (-1)^{\ell(w)} |P_0/L_0|^{-\frac{1}{2}} (k+g)^{-\frac{r}{2}} \mathrm{asdim}(V(\varphi^*_{\mathfrak{a}}(\Lambda^w)_{ss} + m_w\delta_{\mathfrak{a}} - \widehat{\rho}_{\mathfrak{a}\sigma}).$$

Since, by definition, $\mathrm{asdim}(V(\varphi^*_{\mathfrak{a}}(\Lambda^w)_{ss} + m_w\delta_{\mathfrak{a}} - \widehat{\rho}_{\mathfrak{a}\sigma})) = \mathrm{asdim}(V(\varphi^*_{\mathfrak{a}}(\Lambda^w) - \widehat{\rho}_{\mathfrak{a}\sigma}))$, simplifying the constant we are done. $\square$

Turning to the finite dimensional case, if $\mathfrak{a}$ is an equal rank reductive subalgebra of a finite dimensional reductive Lie algebra $\mathfrak{g}$, then (5.8) can be extended to $q$-dimensions. If $V$ is a finite dimensional representation of $\mathfrak{a}$ and $ch(V)$ is its character, then the $q$-dimension of $V$ is

$$\dim_q V = ch(V)(e^{r^\vee_{\mathfrak{a}}}), \tag{5.17}$$

where $r^\vee_{\mathfrak{a}}$ is an element of $\mathfrak{h}$ such that $\alpha(r^\vee_{\mathfrak{a}}) = 1$ for any simple root $\alpha$ of $\mathfrak{a}$. Note that $r^\vee_{\mathfrak{a}}$ coincides with $\rho^\vee_{\mathfrak{a}}$ (the half sum of the positive coroots of $\mathfrak{a}$) if $\mathfrak{a}$ is semisimple. In this case formula (1.2) gives an explicit expression for $\dim_q V$. For a general reductive $\mathfrak{a}$ the element $r^\vee_{\mathfrak{a}}$ is not unique and the definition of $q$-dimension depends on this choice. We shall choose $r^\vee_{\mathfrak{a}}$ as in the following lemma.

**Lemma 5.8.** *Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{a}$ be a reductive equal rank subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}$ be a common Cartan subalgebra and let $\Delta$ (resp $\Delta_{\mathfrak{a}}$) be the set of roots of $\mathfrak{g}$ (resp. $\mathfrak{a}$). Then there exists an element $r \in \mathfrak{h}$, for which $\alpha(r) = 1$ for each simple root $\alpha$ of $\mathfrak{a}$ and $\beta(r) \in \mathbb{Z}$ for some $\beta \in \Delta \setminus \Delta_{\mathfrak{a}}$.*

*Proof.* Assume $\mathfrak{a}$ is semisimple. Then we have to show that $\alpha(\rho^\vee_{\mathfrak{a}}) \in \mathbb{Z}$ for some $\alpha$ in $\Delta \setminus \Delta_{\mathfrak{a}}$. Let $\alpha_1, \dots, \alpha_n$ be a set of simple roots for $\Delta$ and let $\theta = \sum_{i=1}^n a_i\alpha_i$ be the highest root. Fix an index $i_0$, $1 \le i_0 \le n$; then the set $\{-\theta, \alpha_i, \dots, \alpha_{i_0-1}, \alpha_{i_0+1}, \dots, \alpha_n\}$ is a set of simple roots of a semisimple subalgebra $\mathfrak{a}^{i_0}$ of $\mathfrak{g}$. It is a Theorem of Dynkin and Borel – de Siebenthal that any semisimple $\mathfrak{a}$ can be obtained in this way by repeating the procedure several times, and the maximal (equal rank) subalgebras are exactly the $\mathfrak{a}^{i_0}$ with $a_{i_0}$ a prime number. We may assume that $\mathfrak{a} = \mathfrak{a}^{i_0}$ with $a_{i_0}$ prime. Then we have

$$\rho^\vee_{\mathfrak{a}}(\alpha_{i_0}) = 1 - \frac{h}{a_{i_0}},$$

where $h = 1 + \sum_{i=1}^n a_i$ is the Coxeter number. Since $a_{i_0}$ always divides $h$ if it is prime, we can take $\alpha = \alpha_{i_0}$.

If $\mathfrak{a}$ is reductive, start from $\rho^\vee_{\mathfrak{a}} + th$ with $t \in \mathbb{C}$ and $h$ in the center of $\mathfrak{a}$, and choose $t$ is such a way that $\rho^\vee_{\mathfrak{a}} + th$ is integer valued on some root in $\Delta \setminus \Delta_{\mathfrak{a}}$. $\square$

Now we can prove the following

**Proposition 5.9.** *Let $L$ be a finite dimensional irreducible module with highest weight $\lambda$ over a reductive Lie algebra $\mathfrak{g}$. Let $\mathfrak{a}$ be a reductive subalgebra of $\mathfrak{g}$, and $r = r^\vee_{\mathfrak{a}}$ be as in Lemma 5.8. Let $V_w$ denote the irreducible $\mathfrak{a}$-module with highest weight $w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{a}}$, where $w \in W'$ and $W'$ is as in* (1.1). *Then*

$$\sum_{w \in W'} (-1)^{\ell(w)} \dim_q V_w = 0. \tag{5.18}$$

*Proof.* Let $F^+$ and $F^-$ be the even and odd components of the spin representation of $\mathfrak{p}$. (Recall that, since $\mathfrak{a}$ is equal rank in $\mathfrak{g}$, $\mathfrak{p}$ is even dimensional). Then, by (1.1),

$$ch(L \otimes F^+) - ch(L \otimes F^-) = \sum_{w \in W'} (-1)^{\ell(w)} ch(V_w),$$

hence, to prove our claim, we need only to check that

$$(ch(F^+) - ch(F^-))(e^{r^\vee_{\mathfrak{a}}}) = 0.$$

Let $\Delta$ be the set of roots of $\mathfrak{g}$ and choose a positive set of roots $\Delta^+$ for $\mathfrak{g}$. Then $\Delta$ is the disjoint union of $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{p}}$, where $\Delta_{\mathfrak{a}}$ (resp. $\Delta_{\mathfrak{p}}$) is the set of roots in $\Delta$ such that $\mathfrak{g}_\alpha \subset \mathfrak{a}$ (resp. $\mathfrak{g}_\alpha \subset \mathfrak{p}$). Let $\Delta^+_{\mathfrak{p}} = \Delta^+ \cap \Delta_{\mathfrak{p}}$ and set $\mathfrak{p}^\pm = \sum_{\pm\alpha \in \Delta^+_{\mathfrak{p}}} \mathfrak{g}_\alpha$. If $\alpha \in \Delta_{\mathfrak{p}}$, choose $x_\alpha \in \mathfrak{g}_\alpha$ in such a way that $(x_\alpha, x_\beta) = \delta_{\alpha,-\beta}$. Define $E_{\alpha,\beta} \in End(\mathfrak{p})$ by $E_{\alpha,\beta}(x_\gamma) = \delta_{\alpha,\gamma} x_\beta$. A Cartan subalgebra of $so(\mathfrak{p})$ is $\mathfrak{h}_{so(\mathfrak{p})} = \sum_{\alpha \in \Delta^+_{\mathfrak{p}}} (E_{\alpha,\alpha} - E_{-\alpha,-\alpha})$. Define $\epsilon_\alpha \in (\mathfrak{h}_{so(\mathfrak{p})})^*$ by $\epsilon_\alpha(E_{\beta,\beta} - E_{-\beta,-\beta}) = \delta_{\alpha,\beta}$. The $so(\mathfrak{p})$ character of $F^+ - F^-$ is

$$e^{\frac{1}{2}\sum_{\alpha \in \Delta^+_{\mathfrak{p}}} \epsilon_\alpha} \prod_{\alpha \in \Delta^+_{\mathfrak{p}}} (1 - e^{-\epsilon_\alpha}).$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{a}$, hence it embeds in $so(\mathfrak{p})$ via $ad_{|\mathfrak{p}}$. It follows that $h \in \mathfrak{h}$ embeds as $\sum_{\alpha \in \Delta^+_{\mathfrak{p}}} \alpha(h)(E_{\alpha,\alpha} - E_{-\alpha,-\alpha})$. In particular, to check that $ch(F^+ - F^-)(e^{r^\vee_{\mathfrak{a}}}) = 0$, it is enough to find a root $\alpha \in \Delta_{\mathfrak{p}}$ such that $\alpha(r^\vee_{\mathfrak{a}}) = 0$. This root can be found as follows. By Lemma 5.8 we have that $\alpha(r^\vee_{\mathfrak{a}}) \in \mathbb{Z}$ for some $\alpha \in \Delta_{\mathfrak{p}}$. We may assume that $\alpha(r^\vee_{\mathfrak{a}}) > 0$. Then there is a simple root $\beta$ in $\Delta^+_{\mathfrak{a}}$ such that $(\alpha, \beta) > 0$, hence $\alpha - \beta$ is a root. Since the pair is reductive $[\mathfrak{a}, \mathfrak{p}] \subset \mathfrak{p}$, hence $\alpha - \beta \in \Delta_{\mathfrak{p}}$. Now $(\alpha - \beta)(r^\vee_{\mathfrak{a}}) = \alpha(r^\vee_{\mathfrak{a}}) - 1$. If $(\alpha - \beta)(r^\vee_{\mathfrak{a}}) = 0$ we are done. Otherwise repeat the argument substituting $\alpha$ with $\alpha - \beta$. $\square$

# 6 Interlude: the very strange formula

We want to show that formula (4.21) becomes the "very strange formula" (cf. [8, (13.15.4)]) when $\sigma$ is an automorphism of order $m$ (hence it affords a generalization of it, holding for infinite order automorphisms too). Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let $\Pi = \{\eta_1, \dots, \eta_n\}$ be a set of simple roots for $\mathfrak{g}$, $\Delta^+$ the associated subset of positive roots and $\rho$ the corresponding half sum of positive roots.

**Proposition 6.1.** *Let $\kappa(\cdot,\cdot)$ denote the Killing form on $\mathfrak{g}$ and $\sigma$ be an automorphism of order $m$ of type $(s_0, s_1, \dots, s_n; 1)$ ([8, Chapter 8]). Let $\mathfrak{g} = \oplus_{\bar{j}\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}^{\bar{j}}$ be the eigenspace decomposition with respect to $\sigma$. Define $\lambda_s \in \mathfrak{h}^*$ by $\kappa(\lambda_s, \eta_i) = \frac{s_i}{2m}$, $1 \le i \le n$. Then*

$$\kappa(\rho - \lambda_s, \rho - \lambda_s) = \frac{\dim \mathfrak{g}}{24} - \frac{1}{4m^2} \sum_{j=1}^{m-1} j(m-j) \dim \mathfrak{g}^{\bar{j}}. \tag{6.1}$$

*Proof.* Plug $\mathfrak{a} = 0$ in formula (4.21) and choose $\kappa(\cdot,\cdot)$ as invariant form on $\mathfrak{g}$. Then we have $g = \frac{1}{2}$ and the r.h.s. of (4.21) becomes the r.h.s. of (6.1). Let $\theta$ denote the highest root of $\Delta$. We claim now that if we choose as simple roots in $\mathfrak{g}^{\bar{0}}$ the set

$$\Pi_0 = \begin{cases} \{\eta_i \mid s_i = 0\} \cup \{-\theta\} & \text{if } s_0 = 0, \\ \{\eta_i \mid s_i = 0\} & \text{otherwise.} \end{cases}$$

then $\lambda_s = \rho - \rho_\sigma$. To prove this we use the following fact, which is not difficult to prove: if we put $\beta_0 = \frac{s_0}{m}\delta - \theta$, $\beta_i = \frac{s_i}{m}\delta + \eta_i$, then $\{\beta_0, \dots, \beta_n\}$ is a set of simple roots for $\widehat{L}(\mathfrak{g}, \sigma)$. Note that $\overline{\beta_i} = \eta_i$; using (5.4) in our context we have

$$\kappa(\rho - \rho_\sigma, \eta_i) = \frac{\kappa(\eta_i, \eta_i)}{2} - \frac{\kappa(\eta_i, \eta_i)}{2} + \frac{s_i}{2m}$$

as desired. □

# 7 Non-vanishing Dirac cohomology

If $M$ is a highest weight module of $\widehat{L}(\mathfrak{g}, \sigma)$ and $N' = M \otimes F^{\bar{\tau}}(\bar{\mathfrak{p}})$, the Dirac cohomology of the affine Dirac operator is

$$H((G_{\mathfrak{g},\mathfrak{a}})_0^{N'}) = Ker(G_{\mathfrak{g},\mathfrak{a}})_0^{N'} / (Im(G_{\mathfrak{g},\mathfrak{a}})_0^{N'} \cap Ker(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}). \tag{7.1}$$

In this section we obtain a non-vanishing result for $H((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})$ similar to Theorem 3.15 of [13]. Although the general strategy is parallel to Kostant's

one, we obtain the crucial result (Lemma 7.1) using vertex algebra techniques. The setting is as in Section 4; in particular, when we consider the module $L(\Lambda)\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$, the highest weight $\Lambda$ is not assumed to be dominant integral. Note that if $\Lambda$ is dominant integral we have already determined the cohomology: in this case $N'=X=L(\Lambda)\otimes F^{\bar\tau}(\bar{\mathfrak{p}})$ and $H((G_{\mathfrak{g},\mathfrak{a}})_0^X)=Ker(G_{\mathfrak{g},\mathfrak{a}})_0^X$. Indeed, by Proposition 9.2, $Im(G_{\mathfrak{g},\mathfrak{a}})_0^X\cap Ker(G_{\mathfrak{g},\mathfrak{a}})_0^X=0$.

Suppose that $M$ is a highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$. Recall that $\mathfrak{n}'_-=\mathfrak{n}_-+\sum_{j<0}t^j\otimes\mathfrak{g}^j\subset\widehat{L}(\mathfrak{g},\sigma)$. We set $\overline{\mathfrak{n}'}_-=t^{-\frac12}\otimes\overline{\mathfrak{n}_-\cap\mathfrak{p}}+\sum_{j<0}t^{j-\frac12}\otimes\overline{\mathfrak{g}^j\cap\mathfrak{p}}\subset L(\overline{\mathfrak{p}},\overline{\tau})$. Set

$$\mathcal{M}=\mathfrak{n}'_-M\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})+M\otimes\overline{\mathfrak{n}'}_-F^{\overline{\tau}}(\overline{\mathfrak{p}}).$$

**Lemma 7.1.**

$$(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(\mathcal{M})\subset\mathcal{M}.$$

*Proof.* We need the following formulae: if $x\in\mathfrak{g}$ then, in $V^{k+g,1}(R^{super})$,

$$[\tilde x_\lambda G_{\mathfrak{g},\mathfrak{a}}]=\sum_i:\widetilde{[x,b_i]}\overline{b}^i:+\lambda\,k\overline{x}_{\mathfrak{p}},$$

while, if $x\in\mathfrak{p}$,

$$[\overline{x}_\lambda G_{\mathfrak{g},\mathfrak{a}}]=\tilde x+\frac12\sum_i:\overline{[x,b_i]}_{\mathfrak{p}}\overline{b}^i:.$$

These formulae are easily computed using Wick formula and the explicit form for $G_{\mathfrak{g},\mathfrak{a}}$ given in Lemma 4.5. Using (3.2), we deduce that, on $M\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$, we have, if $x\in\mathfrak{g}$,

$$[\tilde x_{(n)},(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}]=\sum_i:\widetilde{[x,b_i]}\overline{b}^i:^{\tau}_{(n+\frac12)}+n\,k\,(\overline{x}_{\mathfrak{p}})^{\tau}_{(n-\frac12)}\tag{7.2}$$

and, if $x\in\mathfrak{p}$,

$$[\overline{x}_{(n)},(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}]=\tilde x^{\tau}_{(n+\frac12)}+\frac12\sum_i:\overline{[x,b_i]}_{\mathfrak{p}}\overline{b}^i:^{\tau}_{(n+\frac12)}.\tag{7.3}$$

If we compute explicitly the normal order using (3.4), then

$$\sum_i:\widetilde{[x,b_i]}\overline{b}^i:^{\tau}_{(n+\frac12)}=\sum_{i,r}\widetilde{[x,b_i]}^{\tau}_{(r)}(\overline{b}^i)^{\tau}_{(-r+n-\frac12)}$$

Suppose now that $v=(t^n\otimes x)(w)\otimes u\in\mathcal{M}$ with $t^n\otimes x\in\mathfrak{n}'_-$. Then either $n<0$ or $n=0$ and $x\in\mathfrak{n}_-$.

Since $v=\tilde{x}^{\tau}_{(n)}(w\otimes u)$, we have that

$$(G_{\mathfrak{g},\mathfrak{a}})^{N'}_0(v)=-[\tilde{x}^{\tau}_{(n)},(G_{\mathfrak{g},\mathfrak{a}})^{N'}_0](v)+\tilde{x}_{(n)}(G_{\mathfrak{g},\mathfrak{a}})^{N'}_0(v),$$

so we need only to check that $\sum_{i,r}\widetilde{[x,b_i]}^{\tau}_{(r)}(\overline{b}^i)^{\tau}_{(-r+n-\frac{1}{2})}(w\otimes u)\in\mathcal{M}$ and that $(\overline{x}_{\mathfrak{p}})^{\tau}_{(n-\frac{1}{2})}(w\otimes u)\in\mathcal{M}$. This is obvious if $n<0$. For $n=0$ then $x_{\mathfrak{p}}\in\mathfrak{n}_-\cap\mathfrak{p}$ so $t^{-\frac{1}{2}}\otimes\overline{x}_{\mathfrak{p}}\in\overline{\mathfrak{n}'}_-$. Moreover $t^r\otimes[x,b_i]\in\mathfrak{n}'_-$ if $r<0$ and $t^{-r-\frac{1}{2}}\otimes\overline{b}^i\in\overline{\mathfrak{n}'}_-$ if $r>0$. It remains only to check the term $r=0$. If $b_i\in\mathfrak{n}_-\cap\mathfrak{p}\oplus\mathfrak{h}_{\mathfrak{p}}$ then $[x,b_i]\in\mathfrak{n}_-$, while if $b_i\in\mathfrak{n}\cap\mathfrak{p}$ then $t^{-\frac{1}{2}}\otimes\overline{b}^i\in\overline{\mathfrak{n}'}_-$.

Suppose now $v=w\otimes\overline{x}_{(n)}u$ with either $n<-\frac{1}{2}$ or $n=-\frac{1}{2}$ and $x\in\overline{\mathfrak{n}_-\cap\mathfrak{p}}$. Arguing as above we need only to check that

$$(\tilde{x})^{\tau}_{(n+\frac{1}{2})}(w\otimes u)+\sum_i:\overline{[x,b_i]}_{\mathfrak{p}}\overline{b}^i:^{\tau}_{(n+\frac{1}{2})}(w\otimes u)\in\mathcal{M},$$

or, equivalently, that $\frac{1}{2}\sum_i:\overline{[x,b_i]}_{\mathfrak{p}}\overline{b}^i:^{\tau}_{(n+\frac{1}{2})}(u)\in\overline{\mathfrak{n}'}_-F^{\overline{\tau}}(\mathfrak{p})$.

Write $u=(\Pi_{i=1}^m t^{r_i}\otimes\overline{y}_i)\cdot 1$ with either $r_i<-\frac{1}{2}$ or $r_i=-\frac{1}{2}$ and $y_i\in\mathfrak{n}_-\cap\mathfrak{p}+\mathfrak{h}_{\mathfrak{p}}$. . We introduce the following notation: if $x\in\mathfrak{p}$, write

$$\gamma(x)=\frac{1}{2}\sum_i:\overline{[x,b_i]}_{\mathfrak{p}}\overline{b}^i:,$$

$\{b_i\},\{b^i\}$ being dual bases of $\mathfrak{p}$. If $y\in\mathfrak{p}$, by Wick formula,

$$[\overline{y}_{\lambda}\gamma(x)]=\overline{[y,x]}_{\mathfrak{p}},$$

thus, if $u'=(\Pi_{i=2}^m t^{r_i}\otimes\overline{y}_i)\cdot 1$, then $\gamma(x)^{\tau}_{(n+\frac{1}{2})}(u)=(\overline{y})^{\tau}_{(r)}\gamma(x)^{\tau}_{(n+\frac{1}{2})}(u')+(\overline{[y,x]}_{\mathfrak{p}})^{\tau}_{(r+n+\frac{1}{2})}(u')$. Since $[(\overline{y})^{\tau}_{(r)},(\overline{a})^{\tau}_{(s)}]=0$ if $t^s\otimes\overline{a}\in\overline{\mathfrak{n}'}_-$, we have that $(\overline{y})^{\tau}_{(r)}\mathcal{M}\subset\mathcal{M}$. Hence we are left with checking that $\gamma(x)^{\tau}_{(n+\frac{1}{2})}(u')\in\mathcal{M}$. By an obvious induction on $m$, we reduce ourselves to check that $\gamma(x)^{\tau}_{(n+\frac{1}{2})}(1)\in\mathcal{M}$. Computing explicitly the normal order and using the fact that, if $\{b_i\}$ is a basis of $\mathfrak{g}^j\cap\mathfrak{p}$ then $\sum_i([x,b_i],b^i)=0$ if $x\in\mathfrak{n}_-\cap\mathfrak{p}$, we find that

$$\begin{aligned}\sum_i:(\overline{[x,b_i]}_{\mathfrak{p}})\overline{b}^i:^{\tau}_{(n+\frac{1}{2})}&=\sum_{i,r<n}(\overline{[x,b_i]}_{\mathfrak{p}})^{\tau}_{(r)}(\overline{b}^i)^{\tau}_{(-r+n-\frac{1}{2})}\\&-\sum_{i,r\geq n}(\overline{b}^i)^{\tau}_{(-r+n-\frac{1}{2})}(\overline{[x,b_i]}_{\mathfrak{p}})^{\tau}_{(r)}.\end{aligned}$$

Thus $\sum_i:(\overline{[x,b_i]}_{\mathfrak{p}})\overline{b}^i:^{\tau}_{(n+\frac{1}{2})}(1)=-\sum_{i,r\geq n}(\overline{b}^i)^{\tau}_{(-r+n-\frac{1}{2})}(\overline{[x,b_i]}_{\mathfrak{p}})^{\tau}_{(r)}(1)$. The terms that are not obviously in $\mathcal{M}$ are those with $n=r$ and $b^i\in\mathfrak{n}\cap\mathfrak{p}+\mathfrak{h}_{\mathfrak{p}}$. But in this case $[x,b_i]_{\mathfrak{p}}\in\mathfrak{n}_-\cap\mathfrak{p}$ and we are done. $\square$

**Corollary 7.2.** *Fix $\Lambda \in \widehat{\mathfrak{h}}_0^*$ and let $M$ be a highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$ of highest weight $\Lambda$. Set $N' = L(\Lambda) \otimes F^{\overline{\tau}}(\bar{\mathfrak{p}})$. If $(\Lambda+\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}} = 0$ then $H((G_{\mathfrak{g},\mathfrak{a}})_0^{N'}) \neq 0$.*

*Proof.* From the explicit description of the basis of $F^{\overline{\tau}}(\overline{\mathfrak{p}})$ given in Lemma 5.1, we see that $F^{\overline{\tau}}(\overline{\mathfrak{p}}) = \mathcal{M} \oplus (v_\Lambda \otimes Cl(t^{-\frac{1}{2}} \otimes \overline{\mathfrak{h}}_\mathfrak{p}) \cdot 1)$. To conclude the proof we need only to show that $(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(v_\Lambda \otimes Cl(t^{-\frac{1}{2}} \otimes \overline{\mathfrak{h}}_\mathfrak{p}) \cdot 1) = 0$. For this, it suffices to show that

$$(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(\overline{h}_{i_1})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1) = 0$$

with $h_{i_j}$ chosen as in (5.2). We prove this by induction on $r$. If $r = 0$, then, by Proposition 4.3 and our assumption that $(\Lambda+\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}} = 0$, we have $(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(v_\Lambda \otimes 1) = 0$. If $r > 0$ then, by the induction hypothesis and (7.3),

$$\begin{aligned}(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(\overline{h}_{i_1})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1) &= \\ &= ((\tilde{h}_{i_1})^{\overline{\tau}}_{(0)} + \gamma(h_{i_1})^{\tau}_{(0)})(\overline{h}_{i_2})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1).\end{aligned}$$

Since $[\gamma(h_{i_1})_\lambda \overline{h}] = \overline{[h_{i_1}, h]} = 0$ for any $h \in \mathfrak{h}_\mathfrak{p}$, we can rewrite the last formula as

$$\begin{aligned}(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(\overline{h}_{i_1})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1) &= \\ &= \Lambda(h_{i_1})(\overline{h}_{i_2})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1) \\ &+ (\overline{h}_{i_2})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes \gamma(h_{i_1})^{\tau}_{(0)}(1)).\end{aligned}$$

But $\gamma(h)^{\tau}_{(0)}(1) = \rho_\sigma(h)\cdot 1$ (see formula (8.34) further on) whence

$$\begin{aligned}(G_{\mathfrak{g},\mathfrak{a}})_0^{N'}(\overline{h}_{i_1})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1) &= \\ &= (\Lambda+\rho_\sigma)(h_{i_1})\left((\overline{h}_{i_2})^{\overline{\tau}}_{(-\frac{1}{2})}\cdots(\overline{h}_{i_r})^{\overline{\tau}}_{(-\frac{1}{2})}(v_\Lambda \otimes 1)\right) = 0\end{aligned}$$

by our assumption that $(\Lambda+\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}} = 0$. $\square$

# 8 An analogue of a conjecture of Vogan in affine setting

## 8.1 The "Vogan conjecture" and its generalization

Recall from (4.29) the function $\varphi_\mathfrak{a} : \mathfrak{h}_0 \oplus \sum_S \mathbb{C}K_S \oplus \mathbb{C}d_\mathfrak{a} \to \widehat{\mathfrak{h}}_0$ and that, extending functionals by zero on $\mathfrak{h}_\mathfrak{p}$, we can view $\widehat{\mathfrak{h}}_\mathfrak{a}^*$ as a subspace of $(\mathfrak{h}_0 \oplus$

$\sum_S \mathbb{C}K_S \oplus \mathbb{C}d_{\mathfrak{a}})^*$. Recall that the complexified Tits cone of an affine algebra $\widehat{L}(\mathfrak{g},\sigma)$ is the set

$$C_{\mathfrak{g}} = \{\lambda \in \widehat{\mathfrak{h}}_0^* \mid Re\,\lambda(K) > 0\}. \tag{8.1}$$

If $f$ is a function on $C_{\mathfrak{g}}$ we denote by $f_{|\widehat{\mathfrak{h}}_{\mathfrak{a}}^*}$ the function on $\varphi_{\mathfrak{a}}^*(C_{\mathfrak{g}}) \cap \widehat{\mathfrak{h}}_{\mathfrak{a}}^*$ defined by $f_{|\widehat{\mathfrak{h}}_{\mathfrak{a}}^*}(\lambda) = (f \circ (\varphi_{\mathfrak{a}}^*)^{-1})(\lambda)$. Suppose that $M$ is a highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$ with highest weight $\Lambda \in C_{\mathfrak{g}}$. We already observed in § 4 that, given $\nu \in (\widehat{\mathfrak{h}}_{\mathfrak{a}})^*$ such that $(M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}))_\nu \neq 0$, then $\nu + \widehat{\rho}_{\mathfrak{a}\sigma} \in \varphi_{\mathfrak{a}}^*(C_{\mathfrak{g}})$ so $f_{|\widehat{\mathfrak{h}}_{\mathfrak{a}}^*}(\nu + \widehat{\rho}_{\mathfrak{a}\sigma})$ makes sense.

The following result is our affine analog of Vogan's conjecture. It appears, in a slightly different formulation, as Theorem 1.1 in the Introduction. In order to use Kac's results from [7] on the holomorphic center of a suitable completion of $U(\widehat{L}(\mathfrak{g},\sigma))$, we need a technical hypothesis on the pair $(\mathfrak{g},\mathfrak{a})$ (which is satisfied in important cases, e.g. when $\mathfrak{a}^{\overline{0}}$ is an equal rank subalgebra of $\mathfrak{g}^{\overline{0}}$ or the set of fixed points of a diagonalizable automorphism of $\mathfrak{g}$).

**Theorem 8.1.** *Assume that the centralizer $C(\mathfrak{h}_{\mathfrak{a}})$ of $\mathfrak{h}_{\mathfrak{a}}$ in $\mathfrak{g}^{\bar{0}}$ equals $\mathfrak{h}_0$. Fix $\Lambda \in \widehat{\mathfrak{h}}_0^*$ such that $\Lambda + \widehat{\rho}_\sigma \in C_{\mathfrak{g}}$ and let $M$ be a highest weight module for $\widehat{L}(\mathfrak{g},\sigma)$ with highest weight $\Lambda$. Let $f$ be a holomorphic $\widehat{W}$-invariant function on $C_{\mathfrak{g}}$. Suppose that a highest weight $\widehat{L}(\mathfrak{a},\sigma)$-module $Y$ of highest weight $\mu$ occurs in the Dirac cohomology $H((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})$ of $N' = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$. Then*

$$f_{|\widehat{\mathfrak{h}}_{\mathfrak{a}}^*}(\mu + \widehat{\rho}_{\mathfrak{a}\sigma}) = f(\Lambda + \widehat{\rho}_\sigma). \tag{8.2}$$

**Remark 8.1.** Assume that $\mathfrak{a}^{\bar{0}}$ is an equal rank subalgebra of $\mathfrak{g}^{\bar{0}}$ and that $\Lambda$ is dominant integral. Then (8.2) is a fairly trivial consequence of Theorem 5.4. Indeed, in the integral dominant case, we have $H((G_{\mathfrak{g},\mathfrak{a}})_0^X) = Ker((G_{\mathfrak{g},\mathfrak{a}})_0^X)$, and formula (5.5) tells us that the only weights appearing in the decomposition are precisely those in the orbit of $\Lambda + \widehat{\rho}_\sigma$. Since $f$ is $\widehat{W}$-invariant, the claim follows.

The following sections are devoted to the proof of Theorem 8.1.

## 8.2 The basic setup

Consider the algebra $\mathcal{W} = U(L'(\mathfrak{g},\sigma)) \otimes Cl(L(\overline{\mathfrak{g}},\overline{\tau}))$ and its quotient algebra $\mathcal{W}^k = \mathcal{W}/\mathcal{W}(K-k)$. We view $L(\mathfrak{g},\sigma) \oplus L(\bar{\mathfrak{g}},\bar{\tau})$ as a subspace of $L'(\mathfrak{g},\sigma)) \oplus L(\overline{\mathfrak{g}},\overline{\tau})$ and identify it with its image in $\mathcal{W}^k$.

Assume that the centralizer of $\mathfrak{h}_{\mathfrak{a}}$ in $\mathfrak{g}^{\bar{0}}$ is the Cartan subalgebra $\mathfrak{h}_0$. Therefore we can fix $f_{\mathfrak{a}} \in \mathfrak{h}_{\mathfrak{a}}$ such that the centralizer of $f_{\mathfrak{a}}$ in $\mathfrak{g}^{\bar{0}}$ is $\mathfrak{h}_0$. We can choose

$\Delta_0^+$ so that $\alpha(f_\mathfrak{a}) > 0$ if $\alpha \in \Delta_0^+$. Choose $c \in \mathbb{R}$ such that $\alpha_i(cd + f_\mathfrak{a}) > 0$ for all $i$. Set $\hat{f}_\mathfrak{a} = cd + f_\mathfrak{a}$. With this choice, we have that if $\alpha \in \widehat{\Delta}^+$, then $\alpha(\hat{f}_\mathfrak{a}) > 0$.

We define a $\mathbb{R}$-grading deg on $L'(\mathfrak{g}, \sigma)$ and on $L(\bar{\mathfrak{g}}, \bar{\tau})$ by setting, for $x \in \mathfrak{g}_\alpha^{\bar{r}}$,

$$\deg(t^r \otimes x) = \deg(t^{r-\frac{1}{2}} \otimes x) = (r\delta + \alpha)(\hat{f}_\mathfrak{a}), \quad \deg(K) = 0 \tag{8.3}$$

and write $L'(\mathfrak{g}, \sigma) = \oplus_j L'(\mathfrak{g}, \sigma)_j$, $L(\bar{\mathfrak{g}}, \bar{\tau}) = \oplus_j L(\bar{\mathfrak{g}}, \bar{\tau})_j$ for the corresponding decomposition into homogeneous components.

We can extend deg to $\mathcal{W}$ thus defining a $\mathbb{R}$-grading $\mathcal{W} = \oplus_j \mathcal{W}_j$. Since $K - k$ is homogeneous, deg induces a $\mathbb{R}$-grading on $\mathcal{W}^k$ that we again denote by deg.

Observe that $\mathfrak{g}^{\bar{r}} = \oplus_{\mu \in \mathfrak{h}_\mathfrak{a}} \mathfrak{g}^{\mu,\bar{r}}$ where $\mathfrak{g}^{\mu,\bar{r}}$ is the $\mathfrak{h}_\mathfrak{a}$-weight space of weight $\mu$. Note that $\mathfrak{g}^{\mu,\bar{r}}$ is homogeneous with respect to deg and $\mathfrak{g}^{\mu,\bar{r}} = \mathfrak{a}^{\mu,\bar{r}} \oplus \mathfrak{p}^{\mu,\bar{r}}$, where $\mathfrak{a}^{\mu,\bar{r}} = \mathfrak{g}^{\mu,\bar{r}} \cap \mathfrak{a}$ and $\mathfrak{p}^{\mu,\bar{r}} = \mathfrak{g}^{\mu,\bar{r}} \cap \mathfrak{p}$. It follows that $L(\mathfrak{a}, \sigma)$, $L(\mathfrak{p}, \sigma)$, $L(\bar{\mathfrak{a}}, \bar{\tau})$, and $L(\bar{\mathfrak{p}}, \bar{\tau})$ all inherit a grading deg from the grading deg on $L(\mathfrak{g}, \sigma)$, $L(\bar{\mathfrak{g}}, \bar{\tau})$ and

$$L(\mathfrak{g}, \sigma)_j = L(\mathfrak{a}, \sigma)_j \oplus L(\mathfrak{p}, \sigma)_j \quad L(\bar{\mathfrak{g}}, \bar{\tau})_j = L(\bar{\mathfrak{a}}, \bar{\tau})_j \oplus L(\bar{\mathfrak{p}}, \bar{\tau})_j. \tag{8.4}$$

Recall the triangular decompositions $L'(\mathfrak{g}, \sigma) = \mathfrak{n}'_- \oplus \mathfrak{h}'_0 \oplus \mathfrak{n}'$ and $L(\overline{\mathfrak{g}}, \overline{\tau}) = \overline{\mathfrak{n}}'_- \oplus \overline{\mathfrak{h}}_0 \oplus \overline{\mathfrak{n}}'$. Set

$$\mathcal{W}^+ = U(\mathfrak{n}') \otimes Cl(\overline{\mathfrak{n}}')$$

and observe that $\mathcal{W}^+ = \bigoplus_{n \in \mathbb{R}^+} \mathcal{W}_n^+$.

Let $\mathcal{F}$ be the algebra of holomorphic functions on $(\mathfrak{h}_0^* \oplus \mathbb{C}\delta) \times (\mathfrak{h}_\mathfrak{a}^* \oplus \mathbb{C}\delta_\mathfrak{a})$. Observe that one can define a product on $\mathcal{W} \otimes \mathcal{F}$ by setting, for $x \in \mathfrak{g}_\alpha^{\overline{r}}$, $y \in \mathfrak{p}^{\mu,\overline{r}}$, $z \in \mathfrak{a}^{\mu,\overline{r}}$, and $f, g \in \mathcal{F}$,

$$f(t^r \otimes x) = (t^r \otimes x) f_{\alpha + r\delta, \alpha_{|\mathfrak{h}_\mathfrak{a}} + r\delta_\mathfrak{a}}, \quad f(t^{r-\frac{1}{2}} \otimes \overline{y}) = (t^{r-\frac{1}{2}} \otimes \overline{y}) f_{0, \mu + r\delta_\mathfrak{a}} \tag{8.5}$$

and

$$[(t^{r-\frac{1}{2}} \otimes \overline{z}), f] = [f, g] = [f, K] = 0, \tag{8.6}$$

where $f_{\alpha,\beta}(\lambda, \mu) = f(\lambda + \alpha, \mu + \beta)$. We will refer to $f_{\alpha,\beta}$ as a *translate* of $f$. Since $(1 \otimes f)\mathcal{W}(K - k) \subset \mathcal{W}(K - k) \otimes \mathcal{F}$ the product on $\mathcal{W} \otimes \mathcal{F}$ factors to define a product on $\mathcal{W}^k \otimes \mathcal{F}$.

Set $j_0 = 0$ and define recursively, for $N \in \mathbb{N}$,

$$j_N = \min(j > j_{N-1} \mid \mathcal{W}_j^+ \neq \{0\}). \tag{8.7}$$

To see that $j_N$ is well defined we argue as follows: if $x \in \mathcal{W}_j^+$ then $x$ is a sum of products $x^1 \cdots x^m$ with $x^i = t^{r_i} \otimes y^i$ or $x^i = t^{r_i - \frac{1}{2}} \otimes \overline{y}^i$ with $y^i \in \mathfrak{g}_{\gamma_i}^{\bar{r}_i}$. Then

$j = \lambda(\hat{f}_{\mathfrak{a}})$ where $\lambda = \sum(r_i\delta + \gamma_i)$. Since $\lambda = \sum n_i\alpha_i$ with $n_i \in \mathbb{N}$ and since $\alpha_i(\hat{f}_{\mathfrak{a}}) \geq 0$, we see that, fixing $R > 0$, then the set $\{j < R \mid \mathcal{W}_j^+ \neq \{0\}\}$ is finite. This in turn implies that $\{j > j_{N-1} \mid \mathcal{W}_j^+ \neq \{0\}\}$ has a minimum.

If $N \in \mathbb{N}$ set

$$F^N = \sum_{n\in\mathbb{R},\, n\geq j_N} \mathcal{W}_n^+.$$

The set $\{(\mathcal{W}^k \otimes \mathcal{F})(F^N \otimes \mathcal{F})\}_{N\in\mathbb{N}}$ is, by PBW theorem, a fundamental system of neighborhoods of 0 in $\mathcal{W}^k \otimes \mathcal{F}$. Let $(\mathcal{W}^k)^{com}_{\mathcal{F}}$ be the corresponding completion. Set

$$\overline{\mathcal{W}^k_{\mathcal{F}}} = \oplus_j \overline{\mathcal{W}^k_j \otimes \mathcal{F}}.$$

The product we just defined on $\mathcal{W}^k \otimes \mathcal{F}$ can be extended to a product on $\overline{\mathcal{W}^k_{\mathcal{F}}}$. Indeed, suppose that $u \in \overline{\mathcal{W}^k_i \otimes \mathcal{F}}$ and $w \in \overline{\mathcal{W}^k_j \otimes \mathcal{F}}$. Writing explicitly $u, w$ as a sequence $u^n \otimes f_n$ and $w^n \otimes g_n$, we now check that $(u^n \otimes f_n)(w^n \otimes g_n)$ converges (i.e. is a Cauchy sequence). We need to check that for each $N$ there is $p$ such that, if $n, m > p$ then $(u^n \otimes f_n)(w^n \otimes g_n) - (u^m \otimes f_m)(w^m \otimes g_m) \in (\mathcal{W}^k \otimes \mathcal{F})(F^N \otimes \mathcal{F})$. Observe that

$$\begin{aligned}(u^n \otimes f_n)(w^n \otimes g_n) - (u^m \otimes f_m)(w^m \otimes g_m) &= (u^n \otimes f_n - u^m \otimes f_m)(w^n \otimes g_n)\\ &\quad + (u^m \otimes f_m)(w^n \otimes g_n - w^m \otimes g_m).\end{aligned}$$

Since $w$ is a Cauchy sequence, there is $N_1$ such that, for $n, m > N_1$, $(w^n \otimes g_n - w^m \otimes g_m) \in (\mathcal{W}^k \otimes \mathcal{F})(F^N \otimes \mathcal{F})$. Since $u$ is a Cauchy sequence, there is $N_2$ such that, for $n, m > N_2$, $(u^n \otimes f_n - u^m \otimes f_m) \in (\mathcal{W}^k \otimes \mathcal{F})(F^M \otimes \mathcal{F})$ with $j_M > j_N - j$. By (8.4) we have that

$$(1 \otimes \mathcal{F})(\mathcal{W}^k_j) = \mathcal{W}^k_j \otimes \mathcal{F}, \tag{8.8}$$

hence $(u^n \otimes f_n - u^m \otimes f_m)(w^m \otimes g_m) \in (\mathcal{W}^k \otimes \mathcal{F})(F^M\mathcal{W}^k_j \otimes \mathcal{F})$. By PBW theorem, $F^M\mathcal{W}^k_j \subset \sum_{\substack{r\leq 0\\ r+s\geq j_N}} \mathcal{W}^k_r\mathcal{W}^+_s \subset \mathcal{W}^k F^N$. Thus $(u^n \otimes f_n - u^m \otimes f_m)(w^m \otimes g_m) \in (\mathcal{W}^k \otimes \mathcal{F})(F^N \otimes \mathcal{F})$, for $n, m > N_2$. Setting $p \geq \max(N_1, N_2)$ we are done.

Note that (8.8) implies that defining $\deg(\mathcal{F}) = 0$ we can extend deg to a grading on $\overline{\mathcal{W}^k_{\mathcal{F}}}$, by declaring $\deg(u) = j$ if $u \in \overline{\mathcal{W}^k_j \otimes \mathcal{F}}$.

Assume that $k + g > 0$ and choose $\Lambda \in \widehat{\mathfrak{h}}_0^*$ a weight of level $k$. If $M$ is a highest weight module for $\widehat{L}(\mathfrak{g}, \sigma)$ of highest weight $\Lambda$, then $N = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ is a representation of $\mathcal{W}^k$. Since, for any $v \in N$, there is $p > 0$ such that $F^M \cdot v = 0$ for $M \geq p$, we can extend the action of $\mathcal{W}^k$ to $\overline{\mathcal{W}^k \otimes 1} \subset \overline{\mathcal{W}^k_{\mathcal{F}}}$.

**Lemma 8.2.** *Let $u \in \overline{\mathcal{W}^k \otimes 1}$ be such that, for any highest weight module $M$ of level $k$ and any $v \in N = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$, relation $u \cdot v = 0$ holds. Then $u = 0$.*

The proof is postponed to Section 9.3.

If $\overline{r} \in \mathbb{R}/\mathbb{Z}$ and $r \in \overline{r}$, let ${}^{\overline{r}}V^{k+g,1}(R^{super})$ be the $e^{2\pi ir}$-eigenspace of $\tau$ in $V^{k+g,1}(R^{super})$. Set $H = \frac{1}{k+g}\tilde{L}^{\mathfrak{g}}_{(1)} - L^{\overline{\mathfrak{g}}}_{(1)}$. Since $\tau$ fixes $\frac{1}{k+g}\tilde{L}^{\mathfrak{g}} - L^{\overline{\mathfrak{g}}}$, the $\mathbb{R}/\mathbb{Z}$-grading induced on $V^{k+g,1}(R^{super})$ by $\tau$ is compatible with $H$, i. e. we can write

$$V^{k+g}(R^{super}) = \bigoplus_{\substack{\overline{r}\in\mathbb{R}/\mathbb{Z}\\ \Delta\in\mathbb{R}}} {}^{\overline{r}}V^{k+g,1}(R^{super})[\Delta],$$

where ${}^{\overline{r}}V^{k+g}(R^{super})[\Delta]$ are the joint eigenspaces for $\tau$ and $H$. If $N$ is a $\tau$-twisted representation of $V^{k+g,1}(R^{super})$ and $a \in {}^{\overline{r}}V^{k+g}(R^{super})[\Delta]$ then, for $r \in \overline{r} - \Delta$, we set $a^N_r = a^N_{(r+\Delta-1)}$. If $a \in {}^{\overline{r}}V^{k+g}(R^{super})[\Delta]$ we set $\overline{r}_a = \overline{r}$ and $\Delta_a = \Delta$. $\Delta_a$ is called the conformal weight of $a$.

Set $\sigma_{f_{\mathfrak{a}}} = e^{ad(f_{\mathfrak{a}})}$. As in § 3.3 we can extend $\sigma_{f_{\mathfrak{a}}}$ to an automorphism $\sigma_{f_{\mathfrak{a}}}$ of $V^{k+g,1}(R^{super})$. If $v \in V^{k+g,1}(R^{super})$ is an eigenvector for $\sigma_{f_{\mathfrak{a}}}$ with eigenvalue $s$ we set $\deg(v) = \log(s)$.

Since $f_{\mathfrak{a}} \in \mathfrak{h}_0$, $\sigma_{f_{\mathfrak{a}}}$ commutes with $\tau$ and clearly fixes $\frac{1}{k+g}\tilde{L}^{\mathfrak{g}} - L^{\overline{\mathfrak{g}}}$, thus ${}^{\overline{r}}V^{k+g}(R^{super})[\Delta]$ is spanned by elements that are homogeneous with respect to deg.

**Lemma 8.3.** *For any $a \in {}^{\overline{r}}V^{k+g}(R^{super})[\Delta]$ and $r \in \overline{r} - \Delta$, there is a unique element $a_r \in \overline{\mathcal{W}^k \otimes 1}$ such that, for any highest weight module $M$ of level $k$ and any $v \in N = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$,*

$$a_r \cdot v = a^N_r \cdot v.$$

*Moreover*

$$\deg(a_r) = cr + \deg(a). \tag{8.9}$$

Uniqueness follows at once from Lemma 8.2; the existence part of the proof as well as the proof of (8.9) is found in Section 9.4.

Recall that, for $x \in \mathfrak{a}$, $\theta(x) = (\tilde{x})_{\mathfrak{a}} - \tilde{x}$.

**Lemma 8.4.** *If $h \in \mathfrak{h}_{\mathfrak{a}}$, we have $\theta(h)_0 - (\rho_\sigma - \rho_{\mathfrak{a}\sigma})(h) \in \overline{\mathcal{W}^k F^1 \otimes 1}$. Moreover, if $D = L^{\overline{\mathfrak{p}}}_0 - (z(\mathfrak{g},\sigma) - z(\mathfrak{a},\sigma) - \frac{1}{16}\dim\mathfrak{p})I$ (cf. (4.30)) then $D \in \overline{\mathcal{W}^k F^1 \otimes 1}$.*

*Proof.* Arguing as in the proof of Lemma 3.4, if $\{x_i\}$ is a basis of $\mathfrak{n}$ and $\{x^i\}$ is the basis of $\mathfrak{n}_-$ dual to $\{x_i\}$, then

$$\theta(h)_0 = \sum_i \overline{[h, x^i]}_0(\overline{x}_i)_0 + (\rho_\sigma - \rho_{\mathfrak{a}\sigma})(h) + u$$

with $u \in \overline{\mathcal{W}^k F^1 \otimes 1}$. We can choose $x_i, x^i$ as root vectors for $\mathfrak{g}^{\bar{0}}$. Recall that we have chosen deg in such a way that $\deg(x_i) \geq 0$ and $\deg(x_i) = 0$ if and only if $x_i \in \mathfrak{n} \cap C(\mathfrak{h}_\mathfrak{a})$. Hence

$$\theta(h)_0 = \sum_{i:x_i \in C(\mathfrak{h}_\mathfrak{a})} \overline{[h, x^i]}_0 (\overline{x}_i)_0 + (\rho_\sigma - \rho_{\mathfrak{a}\sigma})(h) + u'$$

with $u' \in \overline{\mathcal{W}^k F^1 \otimes 1}$. If $\alpha$ is a root that occurs in $C(\mathfrak{h}_\mathfrak{a})$, then $\alpha_{|\mathfrak{h}_\mathfrak{a}} = 0$. Thus

$$\theta(h)_0 = (\rho_\sigma - \rho_{\mathfrak{a}\sigma})(h) + u'$$

as wished.

In order to show that $D \in \overline{\mathcal{W}^k F^1 \otimes 1}$, we first show that there is a constant $d_0 \in \mathbb{C}$ such that $D = d_0 + u$ with $u \in \overline{\mathcal{W}^k F^1 \otimes 1}$. Recall that, up to a constant, $D = L_0^{\bar{\mathfrak{p}}}$. Writing explicitly $L_0^{\bar{\mathfrak{p}}} = -\frac{1}{2}\sum :T(\bar{b}_i)\bar{b}^i:_0$, we see that

$$D = \frac{1}{2}\left(\sum_i (\sum_{r<s_i}(r+\frac{1}{2})(\bar{b}_i)_r \bar{b}^i_{-r} - \sum_{r \geq s_i}(r+\frac{1}{2})\bar{b}^i_{-r}(\bar{b}_i)_r)\right) + const.$$

Choosing $s_i \in [0, 1)$ we see that

$$D = -\frac{1}{4}\left(\sum_i (\sum_{i:s_i=0} \bar{b}^i_0 (\bar{b}_i)_0)\right) + u + const$$

with $u \in \overline{\mathcal{W}^k F^1 \otimes 1}$. We can choose $\{b_i \mid s_i = 0\}$ to be an orthonormal basis of $\mathfrak{p} \cap \mathfrak{g}^{\bar{0}}$ so, since $((\bar{b}_i)_0)^2 = \frac{1}{2}$, $D = u + d_0$ with $d_0 \in \mathbb{C}$ and $u \in \overline{\mathcal{W}^k F^1 \otimes 1}$. To conclude the proof it is enough to recall that the normalization in (4.30) was chosen precisely to obtain that $D^{\bar{\tau}} \cdot 1 = 0$, hence $d_0 = 0$. $\square$

We can define a linear map from $\mathfrak{h}_0 \times \mathfrak{h}_\mathfrak{a} \times \mathbb{C}(d - d_\mathfrak{a})$ to $\overline{\mathcal{W}^k_{\mathcal{F}}}$ by mapping $h \in \mathfrak{h}_0$ to $\tilde{h}_0$, $h \in \mathfrak{h}_\mathfrak{a}$ to $(\tilde{h})_{\mathfrak{a}0}$, and $d_\mathfrak{a} - d$ to $D = L_0^{\bar{\mathfrak{p}}} - (z(\mathfrak{g}, \sigma) - z(\mathfrak{a}, \sigma) - \frac{1}{16}\dim \mathfrak{p})I$ (cf. (4.30)).

Remark that formula (3.1) says that, if $a, b \in V^{k+g,1}(R^{super})$,

$$[a^N_s, b^N_r] = \sum_{j \geq 0} \binom{s + \Delta_a - 1}{j} (a_{(j)}b)^N_{s+r}. \tag{8.10}$$

Hence, by Lemma 8.2,

$$[a_s, b_r] = \sum_{j \geq 0} \binom{s + \Delta_a - 1}{j} (a_{(j)}b)_{s+r}. \tag{8.11}$$

In view of Lemma 3.5 and (8.11), $\tilde{h}_0, (\tilde{h})_{\mathfrak{a}0}$ and $D$ commute with each other, hence we can extend the map defined above to an algebra map $L : S(\mathfrak{h}_0 \times \mathfrak{h}_\mathfrak{a} \times \mathbb{C}(d - d_\mathfrak{a})) \to \overline{\mathcal{W}^k_\mathcal{F}}$.

Clearly we can look upon $S(\mathfrak{h}_0 \times \mathfrak{h}_\mathfrak{a} \times \mathbb{C}(d - d_\mathfrak{a}))$ as a subset of $\mathcal{F}$ by setting $h(\lambda, \mu) = \lambda(h)$ for $h \in \mathfrak{h}_0$, $h(\lambda, \mu) = \mu(h)$ for $h \in \mathfrak{h}_\mathfrak{a}$ and $(d - d_\mathfrak{a})(\lambda, \mu) = \lambda(d) - \mu(d_\mathfrak{a})$. This embedding induces an algebra map $R : S(\mathfrak{h}_0 \times \mathfrak{h}_\mathfrak{a} \times \mathbb{C}(d - d_\mathfrak{a})) \to \overline{\mathcal{W}^k_\mathcal{F}}$.

Let $\mathcal{I}$ be the (two-sided) ideal in $\overline{\mathcal{W}^k_\mathcal{F}}$ generated by $\{L(f) - R(f) \mid f \in S(\mathfrak{h}_0 \times \mathfrak{h}_\mathfrak{a} \times \mathbb{C}(d - d_\mathfrak{a}))\}$. We need to describe the ideal $\mathcal{I}$ more carefully. Note that the product in $\mathcal{W}^k_\mathcal{F}$ has been devised in such a way that, if $x \in \mathfrak{g}^{\overline{r}}_\alpha$, then

$$\tilde{x}_r(L(f) - R(f)) = (L(f') - R(f'))\tilde{x}_r \tag{8.12}$$

and

$$\overline{x}_r(L(f) - R(f)) = (L(f'') - R(f''))(\overline{x}_\mathfrak{p})_r + (L(f) - R(f))(\overline{x}_\mathfrak{a})_r \tag{8.13}$$

with $f'$ and $f''$ suitable translates of $f$. Moreover, if $u \in U(\mathfrak{h}_0) \otimes Cl(\bar{\mathfrak{h}}_0) \otimes \mathcal{F}$,

$$[u, L(f) - R(f)] = 0. \tag{8.14}$$

Observe that (8.12) and (8.13) imply that, if $u \in \mathcal{W}^+$, then $u(L(f) - R(f) = \sum_i (L(f_i) - R(f_i))u_i$ with $u_i \in \mathcal{W}^+$. Since $\deg(L(f)) = \deg(R(f)) = 0$, if $u \in F^N \otimes \mathcal{F}$, then

$$u(L(f) - R(f)) = \sum_i (L(f_i) - R(f_i))u_i \tag{8.15}$$

with $u_i \in F^N \otimes \mathcal{F}$.

It follows that, if $u \in \overline{\mathcal{W}^k_\mathcal{F}}$, then, for each $N \in \mathbb{N}$, we can find $u' \in \mathcal{W}^k_\mathcal{F}$ such that

$$u(L(f) - R(f)) = u'(L(f) - R(f)) + w$$

with $w \in \overline{\mathcal{W}^k_\mathcal{F}(F^N \otimes \mathcal{F})}$. Writing $u' = \sum_i u_i^- u_i^+$ with $u_i^- \in U(\mathfrak{n}'_- \oplus \mathfrak{h}_0) \otimes Cl(\bar{\mathfrak{n}}'_-)$ and $u_i^+ \in (1 \otimes \mathcal{F})U(\mathfrak{n}') \otimes Cl(\bar{\mathfrak{n}}' \oplus \bar{\mathfrak{h}}_0)$, then

$$u(L(f) - R(f)) = \sum_i u_i^-(L(f_i) - R(f_i))u_i^+ + w.$$

Observe finally that (8.12), (8.13), and (8.14) imply that $\mathcal{I}$ is the left ideal generated by $\{L(f) - R(f) \mid f \in S(\mathfrak{h}_0 \times \mathfrak{h}_\mathfrak{a} \times \mathbb{C}(d - d_\mathfrak{a}))\}$. Summarizing, we can conclude that, given $u \in \mathcal{I}$ and $N \in \mathbb{N}$, we can always write $u$ as

$$u = \sum_i u_i^-(L(f_i) - R(f_i))u_i^+ + w \tag{8.16}$$

with $u_i^- \in U(\mathfrak{n}'_- \oplus \mathfrak{h}_0) \otimes Cl(\bar{\mathfrak{n}}'_-)$, $u_i^+ \in (1 \otimes \mathcal{F})U(\mathfrak{n}') \otimes Cl(\bar{\mathfrak{n}}' \oplus \bar{\mathfrak{h}}_0)$, and $w \in \overline{\mathcal{W}^k_\mathcal{F}(F^N \otimes \mathcal{F})}$.

## 8.3 The main filtration

Define

$$\mathcal{A} = \overline{\mathcal{W}^k_{\mathcal{F}}}/\mathcal{I}.$$

By (8.9), $\mathcal{I}$ is generated by homogeneous elements, hence we can define a grading

$$\mathcal{A} = \oplus_j \mathcal{A}_j$$

where $\mathcal{A}_j = \overline{\mathcal{W}^k_j \otimes \mathcal{F}}/(\mathcal{I} \cap \overline{\mathcal{W}^k_j \otimes \mathcal{F}})$. Set, for $p \in \mathbb{N}$

$$\mathcal{A}^p = \left(\overline{\mathcal{W}^k_{\mathcal{F}}(F^p \otimes \mathcal{F})} + \mathcal{I}\right)/\mathcal{I}.$$

We use $\{\mathcal{A}^p\}_{p\in\mathbb{N}}$ as a fundamental system of neighborhoods of 0, and let $\mathcal{A}^{com}$ be the corresponding completion. Set

$$\overline{\mathcal{A}} = \oplus_j(\overline{\mathcal{A}_j}).$$

Similarly to what we did with $\mathcal{W}$, the product on $\mathcal{A}$ can be extended to $\overline{\mathcal{A}}$.

We now start the construction of a sort of PBW-basis for $\mathcal{A}$. If $\{x^1, x^2, \dots\}$, $\{y^1, y^2, \dots\}$ are bases of $\mathfrak{n}'$, $\mathfrak{n}'_-$ respectively and $I = \{i_1, i_2, \dots\}$ is a multi-index with a finite number of non zero elements, we set

$$x^I = (x^1)^{i_1}(x^2)^{i_2}\cdots, \quad y^I = (y^1)^{i_1}(y^2)^{i_2}\cdots. \tag{8.17}$$

Here $x^I, y^I$ are viewed as elements of the symmetric algebra of $L(\mathfrak{g}, \sigma)$.

For $I$ a multi-index with $i_j \le 1$ for all $j$ and a finite number of non zero elements, set

$$\wedge^I x = (x^1)^{i_1} \wedge (x^2)^{i_2}\cdots, \qquad \wedge^I y = (y^1)^{i_1} \wedge (y^2)^{i_2}\cdots.$$

Here $\wedge^I x, \wedge^I y$ are viewed as elements of the exterior algebra $\wedge(L(\mathfrak{g}, \sigma))$.

Finally, fix a basis $\{h^i\}$ of $\mathfrak{h}_0$. If $l = \dim(\mathfrak{h}_0)$ and $S = (s_1, \dots, s_l) \in \mathbb{N}^l$, we set $h^S = (h^1)^{s_1}\cdots(h^l)^{s_l}$ and, if $s_i \le 1$ for all $i$, $(\wedge^S h) = (h^1)^{s_1} \wedge \cdots \wedge (h^l)^{s_l}$. Here $h^S$ is an element of the symmetric algebra $S(\mathfrak{h}_0)$ and $\wedge^S h$ is an element of the exterior algebra $\wedge\mathfrak{h}_0$.

Clearly we can choose $x^i = t^r \otimes x$ with $x \in \mathfrak{g}^{\bar r}$ and $y^i = t^{-r} \otimes y$ with $y \in \mathfrak{g}^{-\bar r}$. Moreover we can assume that $x^i, y^i$ are homogeneous with respect to deg and that $\deg(y^i) = -\deg(x^i)$. If $x^j = t^r \otimes x$ with $x \in \mathfrak{g}^{\bar r}$, then we set $\tilde{x}^j = \tilde{x}_r$ and $\overline{x}^j = \overline{x}_r$. Similarly we define $\tilde{y}^j, \overline{y}^j$. We also set $\overline{h}^i = \overline{h}^i_0$, $\tilde{h}^i = \tilde{h}^i_0$.

For $I$ a multi-index with a finite number of non zero elements and $S \in \mathbb{N}^l$, define, as in (8.17), $\tilde{x}^I = (\tilde{x}^1)^{i_1}(\tilde{x}^2)^{i_2}\cdots$, $\tilde{y}^I = (\tilde{y}^1)^{i_1}(\tilde{y}^2)^{i_2}\cdots$ $\tilde{h}^S = (\tilde{h}^1)^{s_1}\cdots(\tilde{h}^l)^{s_l}$. If $i_j \le 1$, similarly define $\overline{x}^I$, $\overline{y}^I$. If $s_i \le 1$ for $i = 1,\dots,l$, set $\overline{h}^S = (\overline{h}^1)^{s_1}\cdots(\overline{h}^l)^{s_l}$. Here $\tilde{x}^I$, $\tilde{y}^I,\dots$ etc. are seen as elements of $\overline{\mathcal{W}^k_{\mathcal{F}}}$ or of $\overline{\mathcal{A}}$. Finally define

$$\deg(I) = \sum_j i_j \deg(x^j), \qquad |I| = \sum_j i_j.$$

If $u\otimes f \in \mathcal{W}^k_{\mathcal{F}}$ and $a\otimes g \in F^N\otimes\mathcal{F}$, then, by the triangular decomposition, we can write $u$ as a sum of terms of type $u^-u^+$ with $u^- \in U(\mathfrak{n}'_-\oplus\mathfrak{h}_0)\otimes Cl(\bar{\mathfrak{n}}'_-\oplus\bar{\mathfrak{h}}_0)$ and $u^+ \in \mathcal{W}^+$. Note also that, by (8.5), $(1\otimes\mathcal{F})\mathcal{W}^+ = \mathcal{W}^+\otimes\mathcal{F}$, hence, by (8.8), $a\otimes g \in (1\otimes\mathcal{F})F^N$. It follows that we can write $(u\otimes f)(a\otimes g)$ as a sum of terms of type $u^-f'u^+a^+$ with $u^+ \in \mathcal{W}^+$ and $a^+ \in F^N$. By applying PBW-theorem, we can write any element of $\mathcal{W}^k_{\mathcal{F}}(F^N\otimes\mathcal{F})$ as a sum of terms of type

$$\tilde{y}^I\bar{y}^L\tilde{h}^S f\bar{h}^T\tilde{x}^H\bar{x}^K \tag{8.18}$$

with $\deg(H+K) \ge j_N$ and $f \in \mathcal{F}$. Therefore the elements of $\overline{\mathcal{W}^k_{\mathcal{F}}(F^N\otimes\mathcal{F})}$ are series $\sum_{n=N}^\infty u_n$ with $\deg(u_n)$ bounded and $u_n$ a finite sum of monomials as in (8.18) with $\deg(H+K) \ge j_n$.

Using the fact that $\tilde{h}^S = L(h^S)$, we see that an element of $\mathcal{A}^p$ is a series $\sum_{n=p}^\infty u_n$ with $\deg(u_n)$ bounded and $u_n$ a finite sum

$$u_n = \sum_{\substack{I,L,T,H,K\\ \deg(H+K)\ge j_n}} \tilde{y}^I\bar{y}^L f_{I,L,T,H,K}\bar{h}^T\tilde{x}^H\bar{x}^K. \tag{8.19}$$

This writing is, however, not unique. To afford uniqueness we need to restrict the functions to a suitable subspace of $(\mathfrak{h}_0^*\oplus\mathbb{C}\delta)\times(\mathfrak{h}_\mathfrak{a}^*\oplus\mathbb{C}\delta_\mathfrak{a})$. Define

$$\begin{aligned} C_{diag} = \{(\Lambda,\lambda) \in &(\mathfrak{h}_0^*\oplus\mathbb{C}\delta)\times(\mathfrak{h}_\mathfrak{a}^*\oplus\mathbb{C}\delta_\mathfrak{a}),\,|\\ &\qquad(\Lambda+\rho_\sigma)(h) = (\lambda+\rho_{\mathfrak{a}\sigma})(h)\,\forall\, h\in\mathfrak{h}_\mathfrak{a}\oplus\mathbb{C}d_\mathfrak{a}\},\end{aligned}$$

$$I_{diag} = \{f\in\mathcal{F} \mid f_{|C_{diag}} = 0\}.$$

Define $K^p \subset S(\mathfrak{n}'\oplus\mathfrak{n}'_-)\otimes\wedge(\mathfrak{n}'\oplus\mathfrak{n}'_-)$ as

$$K^p = Span\left(y^Ix^H\otimes\wedge^L y\wedge^K x \mid \deg(H+K) = j_p\right)$$

($j_p$ is defined in (8.7)).

The following Lemma gives a sort of PBW-theorem for $\overline{\mathcal{A}}$.

**Lemma 8.5.** *The map*

$$y^I x^H \otimes \wedge^L y \wedge^K x \otimes f \otimes \wedge^T h \mapsto \tilde{y}^I \overline{y}^L f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}}$$

*extends to a linear onto map*

$$\mathcal{S} : K^p \otimes \mathcal{F} \otimes \wedge(\mathfrak{h}_0) \to \overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}. \tag{8.20}$$

*Moreover* $Ker\,\mathcal{S} = K^p \otimes I_{diag} \otimes \wedge(\mathfrak{h}_0)$ *thus* $\mathcal{S}$ *induces a linear isomorphism, still denoted by* $\mathcal{S}$*,*

$$K^p \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge(\mathfrak{h}_0) \longrightarrow \overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}. \tag{8.21}$$

*Proof.* The first statement amounts to proving that the set

$$\{\tilde{y}^I \overline{y}^L f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}} \mid \deg(H+K) = j_p\}$$

spans $\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}$, so assume that $a + \overline{\mathcal{A}^{p+1}} \in \overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}$. We can assume that $a \in \mathcal{A}^p$, hence $a = \sum_{n=p}^{\infty} u_n$ with $u_n$ as in (8.19). Thus

$$a + \overline{\mathcal{A}^{p+1}} = u_p + \overline{\mathcal{A}^{p+1}} = \sum_{\substack{I,L,T,H,K \\ \deg(H+K)=j_p}} \tilde{y}^I \bar{y}^L f_{I,L,T,H,K} \bar{h}^T \tilde{x}^H \bar{x}^K + \overline{\mathcal{A}^{p+1}}.$$

as desired.

In order to prove the second statement suppose that

$$a = \sum \tilde{y}^I \overline{y}^L f_{I,L,T,H,K} \overline{h}^T \tilde{x}^H \overline{x}^H \in \overline{\mathcal{A}^{p+1}}.$$

Then $a = \lim a_i$ with $a_i \in \mathcal{A}^{p+1}$, hence, since $a \in \mathcal{A}$, we have that $a \in \mathcal{A}^{p+1}$. This says that $a \in (\overline{\mathcal{W}^k_{\mathcal{F}}(F^{p+1} \otimes \mathcal{F})} + \mathcal{I})/\mathcal{I}$, hence we can write

$$\tilde{y}^I \bar{y}^L f_{I,L,T,H,K} \bar{h}^T \tilde{x}^H \bar{x}^K = u + w'$$

with $u \in \mathcal{I}$ and $w' \in \overline{\mathcal{W}^k_{\mathcal{F}}(F^{p+1} \otimes \mathcal{F})}$. By (8.16) we can write

$$u = \sum_i u_i^-(L(f_i) - R(f_i))u_i^+ + w$$

with $u_i^- \in U(\mathfrak{n}'_- \oplus \mathfrak{h}_0) \otimes Cl(\bar{\mathfrak{n}}'_-)$, $u_i^+ \in (1 \otimes \mathcal{F})U(\mathfrak{n}') \otimes Cl(\bar{\mathfrak{n}}' \oplus \bar{\mathfrak{h}}_0)$, and $w \in \overline{\mathcal{W}^k_{\mathcal{F}}(F^{p+1} \otimes \mathcal{F})}$. Writing $u_i^-$ as a linear combination of terms $\tilde{y}^I \overline{y}^L \tilde{h}^S$, $u_i^+$ as a linear combination of terms $f_i \overline{h}^T \tilde{x}^H \overline{x}^K$ and setting $w'' = w + w'$ we can write

$$\begin{aligned} &\sum \tilde{y}^I \overline{y}^L f_{I,L,T,H,K} \overline{h}^T \tilde{x}^H \overline{x}^K \\ &\quad = \sum \tilde{y}^I \overline{y}^L \tilde{h}^S (L(p_{I,L,S,T,H,K,i}) - R(p_{I,L,S,T,H,K,i})) f_i \overline{h}^T \tilde{x}^H \overline{x}^K + w'', \end{aligned} \tag{8.22}$$

with $w'' \in \overline{\mathcal{W}^k_{\mathcal{F}}(F^{p+1} \otimes \mathcal{F})}$.

Since $\tilde{h}^S = L(h^S)$, by substituting $\tilde{h}^S(L(p)-R(p))$ with $L(h^S p)-R(h^S p))$ $-(L(h^S) - R(h^S))R(p)$, we can assume that $S = 0$, thus (8.22) simplifies to

$$\begin{aligned}\sum & \tilde{y}^I \overline{y}^L f_{I,L,T,H,K} \overline{h}^T \tilde{x}^H \overline{x}^K \\ &= \sum \tilde{y}^I \overline{y}^L (L(p_{I,K,J,H,S,i}) - R(p_{I,K,J,H,S,i})) f_i \overline{h}^T \tilde{x}^H \overline{x}^K + w''. \end{aligned} \tag{8.23}$$

Fix a basis $\{v_i\}$ of $\mathfrak{h}_{\mathfrak{a}}$. By performing the affine change of variables $(u, v, d - d_{\mathfrak{a}}) \to (u', v', d - d_{\mathfrak{a}})$, where

$$\begin{cases} u' = u & \text{if } u \in \mathfrak{h}_0, \\ v' = (-v, v, 0) - (\rho_\sigma - \rho_{\sigma\mathfrak{a}})(v) & \text{if } v \in \mathfrak{h}_{\mathfrak{a}}, \end{cases}$$

and setting $x_0 = d - d_{\mathfrak{a}}$ and $x_i = v'_i$ for $i > 0$, we can write a polynomial $P \in S(\mathfrak{h}_0 \times \mathfrak{h}_{\mathfrak{a}} \times \mathbb{C}(d - d_{\mathfrak{a}}))$ as

$$P = P^0 + \sum_i x_i P_i \tag{8.24}$$

with $P^0, P_i$ suitable polynomials and $P^0 = P^0(u', 0, 0)$ depending only on $(u', 0, 0)$. By Lemma 8.4, $L(x_i) \in \overline{\mathcal{W}^k_{\mathcal{F}} F^1}$ for all $i$. Hence

$$L(P) = L(P^0) + w^{>0}$$

with $w^{>0} \in \overline{\mathcal{W}^k_{\mathcal{F}} F^1}$. We can therefore write that

$$\begin{aligned}\sum & \tilde{y}^I \overline{y}^L (w^{>0}_{I,L,T,H,K,i}) f_i \overline{h}^T \tilde{x}^H \overline{x}^K + w'' \\ &= \sum \tilde{y}^I \overline{y}^L f_{I,L,T,H,K} \overline{h}^T \tilde{x}^H \overline{x}^K - \sum \tilde{y}^I \overline{y}^L (L(p^0_{I,L,T,H,K,i})) f_i \overline{h}^T \tilde{x}^H \overline{x}^K \\ &+ \sum \tilde{y}^I \overline{y}^L R(p_{I,L,T,H,K,i}) f_i \overline{h}^T \tilde{x}^H \overline{x}^K. \end{aligned}$$

Let $LHS$ denote the left hand side of the above equation and $RHS$ the right hand side. Set $p' = \inf\{q \mid j_q = \deg(H + K),\ p_{I,L,T,H,K,i} \neq 0\}$. If $p' < p$ then $LHS \in \overline{\mathcal{W}^k_{\mathcal{F}}(F^{p'+1})}$. Since $RHS \in \mathcal{W}^k_{\mathcal{F}}$ we have that $RHS \in \mathcal{W}^k_{\mathcal{F}} F^{p'+1}$. If $\deg(H+K) = j_{p'}$, using PBW and comparing terms, we find that $L(p^0_{I,L,T,H,K,i})$ must be a constant. We can clearly assume this constant to be zero, obtaining that $p^0_{I,L,T,H,K,i} = 0$, or, equivalently, that $R(p_{I,L,T,H,K,i}) \in I_{diag}$. Thus, if we set $P = p_{I,L,T,H,K,i}$ in (8.24) we can write $p_{I,L,T,H,K,i} = \sum_j x_j P_j$ with $P_j$ suitable polynomials. Now remark that $L(x_j P_j) - R(x_j P_j) =$

$(L(P_j)-R(P_j))L(x_j)+(L(x_j)-R(x_j))R(P_j)$ and that $L(x_j)\in\overline{\mathcal{W}^k_{\mathcal{F}}F^1}$, so, if $\deg(H+K)=j_{p'}$,

$$\tilde{y}^I\overline{y}^L(L(P)-R(P))f_i\overline{h}^T\tilde{x}^H=\sum_j\tilde{y}^I\overline{y}^L(L(x_j)-R(x_j))f'_j\overline{h}^T\tilde{x}^H$$
$$+\sum_{\deg(H+K)>j_{p'}}\tilde{y}^I\overline{y}^L(L(p_{I,K,J,H,S,i})-R(p_{I,K,J,H,S,i}))f_i\overline{h}^T\tilde{x}^H.$$

Collecting terms, (8.23) gets rewritten as

$$\begin{aligned}&\sum\tilde{y}^I\overline{y}^Kf_{I,K,J,H,S}\overline{h}^S\tilde{x}^J\overline{x}^H\\&\quad=\sum_{\deg(J+K)=j_{p'}}\sum_j\tilde{y}^I\overline{y}^K(L(x_j)-R(x_j))f'_j\overline{h}^S\tilde{x}^J\overline{x}^H\\&\quad+\sum_{\deg(J+K)>j_{p'}}\tilde{y}^I\overline{y}^K(L(p_{I,K,J,H,S,i})-R(p_{I,K,J,H,S,i}))g_i\overline{h}^S\tilde{x}^J\overline{x}^H+w''.\end{aligned}\tag{8.25}$$

Moreover degree considerations imply that we must have that

$$\sum_j R(x_j)f'_j=0.$$

For $j>0$ write $f'_j=p_j+R(x_0)q_j$ with $p_j,q_j\in\mathcal{F}$ and $p_j$ independent of $R(x_0)$. Then $\sum_{j>0}R(x_j)p_j=0$ and $f'_0=-\sum_{i>0}R(x_j)q_j$. Substituting, we find that

$$\begin{aligned}&\sum_j(L(x_j)-R(x_j))f'_j\\&\quad=\sum_{j>0}(L(x_j)-R(x_j))R(x_0)q_j-\sum_{j>0}(L(x_0)-R(x_0))R(x_j)q_j\\&\quad+\sum_{j>0}(L(x_j)-R(x_j))p_j\\&\quad=\sum_{j>0}(L(x_j)R(x_0)-L(x_0)R(x_j))q_j+\sum_{j>0}(L(x_j)-R(x_j))p_j\\&\quad=\sum_{j>0}L(x_0)(L(x_j)-R(x_j))q_j-\sum_{j>0}L(x_j)(L(x_0)-R(x_0))q_j\\&\quad+\sum_{j>0}(L(x_j)-R(x_j))p_j.\end{aligned}$$

Since $L(x_j) \in \overline{\mathcal{W}^k_{\mathcal{F}} F^1}$ we obtain that (8.25) becomes

$$\begin{aligned}\sum & \tilde{y}^I \overline{y}^K f_{I,K,J,H,S} \overline{h}^S \tilde{x}^J \overline{x}^H \\ &= \sum_{\deg(J+K)=j_{p'}} \sum_{j>0} \tilde{y}^I \overline{y}^K (L(x_j) - R(x_j)) p_j \overline{h}^S \tilde{x}^J \overline{x}^H \\ &+ \sum_{\deg(J+K)>j_{p'}} \tilde{y}^I \overline{y}^K (L(p'_{I,K,J,H,S,i}) - R(p'_{I,K,J,H,S,i})) g'_i \overline{h}^S \tilde{x}^J \overline{x}^H + w''.\end{aligned}$$

Repeating the argument for all the variables $x_j$ we can rewrite (8.25) as

$$\begin{aligned}&\sum \tilde{y}^I \overline{y}^K f_{I,K,J,H,S} \overline{h}^S \tilde{x}^J \overline{x}^H \\ &= \sum_{\deg(J+K)>j_{p'}} \tilde{y}^I \overline{y}^K (L(p''_{I,K,J,H,S,i}) - R(p''_{I,K,J,H,S,i})) g''_i \overline{h}^S \tilde{x}^J \overline{x}^H + w''.\end{aligned}$$

We can therefore assume that $p' \geq p$. Again we deduce that $RHS \in \mathcal{W}^k_{\mathcal{F}} F^{p+1}$.

Comparing terms we find that, if $\deg(H+K) = j_p$, then $R(p_{I,K,J,H,S,i}) \in I_{diag}$ and

$$f_{I,K,J,H,S} = \sum_i R(p_{I,K,J,H,S,i}) f_i \in I_{diag}$$

as desired. □

We introduce an increasing filtration on $\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}$ by

$$(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_n = Span\left(\overline{h}^T \tilde{y}^J \overline{y}^R f \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}} \mid |J| + |H| \leq n\right). \tag{8.26}$$

Note that the associated graded vector space is given by

$$Gr(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}) = \bigoplus_n Gr_n(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})$$

where, setting $K^p_n = (S^n(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \wedge(\mathfrak{n}' \oplus \mathfrak{n}'_-)) \cap K^p$,

$$Gr_n(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}) = \mathcal{S}(K^p_n \otimes \mathcal{F} \otimes \wedge \mathfrak{h}_0).$$

**Remark 8.2.** Set $J^p = Span(\overline{y}^J \tilde{y}^I \overline{h}^U \tilde{x}^H \overline{x}^K \mid \deg(H+K) = j_p)$. Lemma 8.5 implies that an element $u$ in the closure of $\sum_p J^p$ can be written uniquely as a series $\sum_{i=0}^{\infty} u_i$ with $u_i \in J^i$. In particular $u \in \overline{\mathcal{A}^p}$ if and only if $u_i = 0$ for $i < p$.

## 8.4 The basic complex

Let $\mathbf{d} : \overline{\mathcal{A}} \to \overline{\mathcal{A}}$ be the superbracket operator $\mathbf{d}(a) = [(G_{\mathfrak{g},\mathfrak{a}})_0, a]$. Set

$$\overline{\mathcal{A}}^{inv} = \{a \in \overline{\mathcal{A}} \mid [d_{\mathfrak{a}}, a] = 0\}.$$

**Lemma 8.6.**

1. *$\overline{\mathcal{A}}^{inv}$ is $\mathbf{d}$-stable.*
2. *$\mathbf{d}^2 = 0$ on $\overline{\mathcal{A}}^{inv}$.*

*Proof.* By (3.12), $[\tilde{L}_0^{\mathfrak{g}}, \tilde{x}_r] = -(k+g)r\tilde{x}_r$ and $[\tilde{L}_0^{\mathfrak{g}}, \overline{x}_r] = 0$. By the definition of the product in $\mathcal{A}$ and (3.10) we see that $[\tilde{L}_0^{\mathfrak{g}}, f] = 0$ for $f$ in $\mathcal{F}$. On the other hand in (8.5) we set $[d, \tilde{x}_r] = r\tilde{x}_r$, $[d, \overline{x}_r] = 0$ and $[d, f] = 0$ for $f$ in $\mathcal{F}$. Thus bracketing with $d$ and $\tilde{L}_0^{\mathfrak{g}}$ stabilizes the subalgebra of $\overline{\mathcal{A}}$ generated by $\tilde{x}_r, \overline{x}_r, f$ and if $x$ is in this subalgebra, then $[\tilde{L}_0^{\mathfrak{g}}, x] = -(k+g)[d, x]$. By Lemma 8.5 this subalgebra is dense in $\overline{\mathcal{A}}$, hence

$$[\tilde{L}_0^{\mathfrak{g}}, x] = -(k+g)[d, x]$$

for all $x \in \overline{\mathcal{A}}$.

To prove the first statement it is enough to show that $(G_{\mathfrak{g},\mathfrak{a}})_0 \in \overline{\mathcal{A}}^{inv}$. To check that $[(G_{\mathfrak{g},\mathfrak{a}})_0, d_{\mathfrak{a}}] = 0$ we recall that $d_{\mathfrak{a}} = d + D$, so $[(G_{\mathfrak{g},\mathfrak{a}})_0, d_{\mathfrak{a}}] = [(G_{\mathfrak{g},\mathfrak{a}})_0, d + L_0^{\overline{\mathfrak{p}}}] = [(G_{\mathfrak{g},\mathfrak{a}})_0, -\frac{1}{k+g}\tilde{L}_0^{\mathfrak{g}} + L_0^{\overline{\mathfrak{p}}}] = [(G_{\mathfrak{g},\mathfrak{a}})_0, -\frac{1}{k+g}(\tilde{L}_0^{\mathfrak{g}} - \tilde{L}_0^{\mathfrak{a}}) + L_0^{\overline{\mathfrak{p}}}]$. By (4.18), $[(G_{\mathfrak{g},\mathfrak{a}})_0, -\frac{1}{k+g}(\tilde{L}_0^{\mathfrak{g}} - \tilde{L}_0^{\mathfrak{a}}) + L_0^{\overline{\mathfrak{p}}}] = [(G_{\mathfrak{g},\mathfrak{a}})_0, -\frac{1}{k+g}(G_{\mathfrak{g},\mathfrak{a}})_0^2] = 0$, $(G_{\mathfrak{g},\mathfrak{a}})_0^2$ being even.

To prove the second statement, we first notice that $\mathbf{d}^2(a) = [(G_{\mathfrak{g},\mathfrak{a}})_0^2, a]$, $(G_{\mathfrak{g},\mathfrak{a}})_0$ being odd. Arguing as above, we see that, if $a \in \overline{\mathcal{A}}^{inv}$, $[(G_{\mathfrak{g},\mathfrak{a}})_0^2, a] = -(k+g)[d_{\mathfrak{a}}, a] = 0$. □

By (8.4) we can find a basis of $\mathfrak{p}$ homogeneous w.r.t. deg. Then (4.22) implies that $(G_{\mathfrak{g},\mathfrak{a}})_0$ has degree 0. It follows that $[(G_{\mathfrak{g},\mathfrak{a}})_0, \overline{\mathcal{A}^p}] \subset \overline{\mathcal{A}^p}$. Thus

$$[(G_{\mathfrak{g},\mathfrak{a}})_0, \overline{\mathcal{A}^p} \cap \overline{\mathcal{A}}^{inv}] \subset \overline{\mathcal{A}^p} \cap \overline{\mathcal{A}}^{inv}.$$

Set $\overline{\mathcal{A}^p_{inv}} = \overline{\mathcal{A}^p} \cap \overline{\mathcal{A}}^{inv}$. We can therefore define a differential

$$\overline{d}_p : \overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}} \to \overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}$$

by

$$\overline{d}_p(x + \overline{\mathcal{A}^{p+1}_{inv}}) = \mathbf{d}(x) + \overline{\mathcal{A}^{p+1}_{inv}}.$$

By (8.4), $[d_{\mathfrak{a}}, \overline{\mathcal{A}^p}] \subset \overline{\mathcal{A}^p}$, hence we can define an action of $d_{\mathfrak{a}}$ on $\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}$. If $x \in \overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}$, let us write $d_{\mathfrak{a}} \cdot x$ for this action. Using the isomorphism $\mathcal{S}$ we can lift this action to $K^p \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$. In order to describe explicitly this action, set

$$\mathfrak{n}'_{\mathfrak{p}} = \mathfrak{n} \cap \mathfrak{p} \oplus \sum_{r>0} t^r \otimes (\mathfrak{g}^{\overline{r}} \cap \mathfrak{p}), \quad \mathfrak{n}'_{\mathfrak{a}} = \mathfrak{n} \cap \mathfrak{a} \oplus \sum_{r>0} t^r \otimes (\mathfrak{g}^{\overline{r}} \cap \mathfrak{a}).$$

Analogously define $(\mathfrak{n}'_-)_{\mathfrak{p}}$ and $(\mathfrak{n}'_-)_{\mathfrak{a}}$. Since $[d_{\mathfrak{a}}, \tilde{x}_r] = r\tilde{x}_r$ for $x \in \mathfrak{g}^{\overline{r}}$, $[d_{\mathfrak{a}}, \overline{x}_r] = r\overline{x}_r$ for $x \in \mathfrak{g}^{\overline{r}} \cap \mathfrak{p}$, $[d_{\mathfrak{a}}, \overline{x}_r] = 0$ for $x \in \mathfrak{g}^{\overline{r}} \cap \mathfrak{a}$, and $[d_{\mathfrak{a}}, f\overline{h}^T] = 0$, we see that $d_{\mathfrak{a}}$ acts on $K^p \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$ as the derivation on $S(\mathfrak{n}' \oplus \mathfrak{n}'_-)$ such that $d_{\mathfrak{a}} \cdot (t^r \otimes x) = r(t^r \otimes x)$ and as the even derivation on $\wedge(\mathfrak{n}'_{\mathfrak{p}} \oplus (\mathfrak{n}'_-)_{\mathfrak{p}})$ such that $d_{\mathfrak{a}} \cdot (t^r \otimes x) = r(t^r \otimes x)$. Moreover $d_{\mathfrak{a}}$ acts trivially on $\wedge(\mathfrak{n}'_{\mathfrak{a}} \oplus (\mathfrak{n}'_-)_{\mathfrak{a}})$ and on $\mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$.

By (8.4), we can choose the basis $\{x^i\}$ as the union of a basis of $\mathfrak{n}'_{\mathfrak{p}}$ with a basis of $\mathfrak{n}'_{\mathfrak{a}}$. Likewise, the basis $\{y^i\}$ can be chosen as the union of bases of $(\mathfrak{n}'_-)_{\mathfrak{p}}$ and $(\mathfrak{n}'_-)_{\mathfrak{a}}$. With this choice of bases, it is clear that the monomials $y^I x^J \otimes \wedge^H y \wedge^K x$ are eigenvectors for the action of $d_{\mathfrak{a}}$. In particular the action of $d_{\mathfrak{a}}$ is semisimple. We can therefore write $K^p = \oplus_s K^{p,s}$ where $K^{p,s}$ denotes the $s$-eigenspace of $K^p$ under the action of $d_{\mathfrak{a}}$. Let $(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{inv}$ be the $d_{\mathfrak{a}}$-invariant subspace of $\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}$. By construction $(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{inv} = \mathcal{S}(K^{p,0} \otimes \mathcal{F} \otimes \wedge\mathfrak{h}_0)$. On the other hand $\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}$ embeds in $(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{inv}$. Moreover, if $y^I x^J \otimes \wedge^H y \wedge^K x \otimes f \otimes \wedge^T h \in K^{p,0} \otimes \mathcal{F} \otimes \wedge\mathfrak{h}_0$, then, clearly, $\overline{y}^I \tilde{y}^H f\overline{h}^T \tilde{x}^J \overline{x}^K$ commutes with $d_{\mathfrak{a}}$, hence the embedding of $\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}$ in $(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{inv}$ is onto. In particular we have an isomorphism

$$\mathcal{S} : K^{p,0} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0 \to \overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}. \tag{8.27}$$

Since $d_{\mathfrak{a}} \cdot S^n(\mathfrak{n}' \oplus \mathfrak{n}'_-) \subset S^n(\mathfrak{n}' \oplus \mathfrak{n}'_-)$ we can write $K^{p,r} = \oplus_n K^{p,r}_n$, where $K^{p,r}_n = K^{p,r} \cap K^p_n$. Set $(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n = (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}) \cap (\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_n$, so the isomorphism in (8.27) gives an isomorphism

$$\oplus_{m \le n}(K^{p,0}_m \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0) \to (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n.$$

Setting $Gr_n(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}) = (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n/(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$, the map $\mathcal{S}$ induces an isomorphism

$$\mathcal{S} : K^{p,0}_n \otimes (\mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0) \to Gr_n(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}), \tag{8.28}$$

and, setting $Gr((\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})) = \oplus_n Gr_n(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})$, an isomorphism

$$\mathcal{S} : K^{p,0} \otimes (\mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0) \to Gr(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}). \tag{8.29}$$

By the decompositions $\mathfrak{n}' = \mathfrak{n}'_\mathfrak{p} \oplus \mathfrak{n}'_\mathfrak{a}$, $\mathfrak{n}'_- = (\mathfrak{n}'_-)_\mathfrak{p} \oplus (\mathfrak{n}'_-)_\mathfrak{a}$, we have

$$S(\mathfrak{n}' \oplus \mathfrak{n}'_-) = S(\mathfrak{n}'_\mathfrak{p} \oplus (\mathfrak{n}'_-)_\mathfrak{p}) \otimes S(\mathfrak{n}'_\mathfrak{a} \oplus (\mathfrak{n}'_-)_\mathfrak{a}),$$
$$\wedge(\mathfrak{n}' \oplus \mathfrak{n}'_-) = \wedge(\mathfrak{n}'_\mathfrak{p} \oplus (\mathfrak{n}'_-)_\mathfrak{p}) \otimes \wedge(\mathfrak{n}'_\mathfrak{a} \oplus (\mathfrak{n}'_-)_\mathfrak{a}).$$

Set

$$K^p_\mathfrak{p} = K^p \cap (S(\mathfrak{n}'_\mathfrak{p} \oplus (\mathfrak{n}'_-)_\mathfrak{p}) \otimes \wedge(\mathfrak{n}'_\mathfrak{p} \oplus (\mathfrak{n}'_-)_\mathfrak{p})),$$
$$K^p_\mathfrak{a} = K^p \cap (S(\mathfrak{n}'_\mathfrak{a} \oplus (\mathfrak{n}'_-)_\mathfrak{a}) \otimes \wedge(\mathfrak{n}'_\mathfrak{a} \oplus (\mathfrak{n}'_-)_\mathfrak{a})),$$

and define $K^{p,s}_\mathfrak{a}$, $K^{p,s}_\mathfrak{p}$, $(K^{p,s}_n)_\mathfrak{a}$, $(K^{p,s}_n)_\mathfrak{p}$ analogously. Since $d_\mathfrak{a} \cdot K^p_\mathfrak{a} \subset K^p_\mathfrak{a}$ and $d_\mathfrak{a} \cdot K^p_\mathfrak{p} \subset K^p_\mathfrak{p}$, we can write

$$K^{p,s} = \sum_{\substack{r+t=p\\a+b=s}} (K^{r,a})_\mathfrak{p} \otimes (K^{t,b})_\mathfrak{a}. \tag{8.30}$$

Let $\partial_{\mathfrak{p}_-}$, $\partial_{\mathfrak{p}_+}$ be the Koszul differentials on $S((\mathfrak{n}'_-)_\mathfrak{p}) \otimes \wedge(\mathfrak{n}'_-)_\mathfrak{p}$ and $S(\mathfrak{n}'_\mathfrak{p}) \otimes \wedge\mathfrak{n}'_\mathfrak{p}$ respectively. Consider the complex $C^u = S((\mathfrak{n}'_-)_\mathfrak{p}) \otimes \wedge^u(\mathfrak{n}'_-)_\mathfrak{p} \otimes S((\mathfrak{n}'_-)_\mathfrak{a})$ endowed with the Koszul differential $\partial_{\mathfrak{p}_-} \otimes I$ (denoted for for shortness $\partial_{\mathfrak{p}_-}$ in the following) and the complex $D^v = (S(\mathfrak{n}'_\mathfrak{p}) \otimes \wedge^v(\mathfrak{n}'_\mathfrak{p})) \otimes \wedge\left((\mathfrak{n}'_-)_\mathfrak{a} \oplus \mathfrak{n}'_\mathfrak{a} \oplus \mathfrak{h}_0\right) \otimes \mathcal{F}_{|C_{diag}} \otimes S((\mathfrak{n}')_\mathfrak{a})$ endowed with the "signed" Koszul differential $\partial^{sign}_{\mathfrak{p}_+}$ defined as $\partial_{\mathfrak{p}_+} \otimes (-1)^j I \otimes I \otimes I \otimes I$ on $(S(\mathfrak{n}'_\mathfrak{p}) \otimes \wedge^v(\mathfrak{n}'_\mathfrak{p})) \otimes \wedge^j\left((\mathfrak{n}'_-)_\mathfrak{a} \oplus \mathfrak{h}_0\right) \otimes \wedge\mathfrak{n}'_\mathfrak{a} \otimes \mathcal{F}_{|C_{diag}} \otimes S((\mathfrak{n}')_\mathfrak{a})$. This endows $S(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \wedge(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0 = C \otimes D$ with the differential $\partial_{\mathfrak{p}_-} \otimes \partial^{sign}_{\mathfrak{p}_+}$. By Künneth formula this differential is exact except in degree zero and

$$H_0(C \otimes D) = S(\mathfrak{n}'_\mathfrak{a} \oplus (\mathfrak{n}'_-)_\mathfrak{a}) \otimes \wedge(\mathfrak{n}'_\mathfrak{a} \oplus (\mathfrak{n}'_-)_\mathfrak{a}) \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0.$$

Since $K^p \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$ and $K^{p,s} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$ are subcomplexes of $S(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \wedge(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$, the differential $\partial_{\mathfrak{p}_-} \otimes \partial^{sign}_{\mathfrak{p}_+}$ restricts to differentials on $K^p \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$, $K^{p,s} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$, which we denote by $\partial_p$, $\partial_{p,s}$ respectively. Since $S(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \wedge(\mathfrak{n}' \oplus \mathfrak{n}'_-) \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0 = \oplus_{p,s}(K^{p,s} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0)$, we have that $\partial_{p,s}$ is exact except in degree zero and, by (8.30),

$$H_0(\partial_{p,s}) = K^{p,s}_\mathfrak{a} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0. \tag{8.31}$$

**Lemma 8.7.** *We have*

$$[(G_{\mathfrak{g},\mathfrak{a}})_0, f] \in \overline{\mathcal{A}^1}. \tag{8.32}$$

*Set* $\overline{h}^{T_j} = \overline{h}_0^{t_1} \cdots \widehat{\overline{h}_0^{t_j}} \cdots \overline{h}_0^{t_k}$. *Then, if* $h^{t_j} \in \mathfrak{h}_\mathfrak{p}$ *for each* $j$,

$$[(G_{\mathfrak{g},\mathfrak{a}})_0, \overline{h}^T] = \sum_j (-1)^j (\tilde{h}_0^{t_j} + \rho_\sigma(h^{t_j}))\overline{h}^{T_j} + \overline{u}. \tag{8.33}$$

*with* $\overline{u} \in \overline{\mathcal{A}^1}$.

*Proof.* The first relation is proven via a direct computation.

We write $(G_{\mathfrak{g},\mathfrak{a}})_0$ explicitly using (4.22) and (3.4). The cubic term turns out to be

$$\sum_{i,j}\left(\sum_{h,k=0}^{\infty}(\overline{[b_i,b_j]_{\mathfrak{p}}}_{s_i+s_j-h-1}\bar{b}^i_{-s_i-k-1}\bar{b}^j_{-s_j+h+k+2}-\right.$$
$$\overline{[b_i,b_j]_{\mathfrak{p}}}_{s_i+s_j-h-1}\bar{b}^j_{-s_j+h-k+1}\bar{b}^i_{-s_i+k})+\sum_{h,k=0}^{\infty}(\bar{b}^i_{-s_i-k-1}\bar{b}^j_{-s_j-h+k+1}\overline{[b_i,b_j]_{\mathfrak{p}}}_{s_i+s_j+h}$$
$$\left.-\bar{b}^j_{-s_j-h-k}\bar{b}^i_{-s_i+k}\overline{[b_i,b_j]_{\mathfrak{p}}}_{s_i+s_j+h})\right)+3\sum_i s_i\overline{[b_i,b^i]_{\mathfrak{p}}}_0.$$

Hence the bracket of $f$ with the cubic term is 0. Let us now consider the quadratic term. This turns out to be

$$\sum_i\sum_{h\in\mathbb{Z}}(\tilde{b}_i)_{s_i-h}\bar{b}^i_{-s_i+h}.$$

If $\deg(\bar{b}^i_{-s_i+h})>0$ then, since $\deg(f)=0$, $\deg([f,\bar{b}^i_{-s_i+h}])>0$. It follows that $[f,(\tilde{b}_i)_{s_i-h}\bar{b}^i_{-s_i+h}]=[f,(\tilde{b}_i)_{s_i-h}]\bar{b}^i_{-s_i+h}+(\tilde{b}_i)_{s_i-h}[f,\bar{b}^i_{-s_i+h}]\in\overline{\mathcal{A}^1}$. If $\deg(\bar{b}^i_{-s_i+h})<0$ then $\deg((\tilde{b}_i)_{s_i-h})>0$, hence $\deg([f,(\tilde{b}_i)_{s_i-h})])>0$ and $[f,(\tilde{b}_i)_{s_i-h}\bar{b}^i_{-s_i+h}]=\bar{b}^i_{-s_i+h}[f,(\tilde{b}_i)_{s_i-h}]+[f,\bar{b}^i_{-s_i+h}](\tilde{b}_i)_{s_i-h}\in\overline{\mathcal{A}^1}$. Finally, if $\deg(\bar{b}^i_{-s_i+h})=0$, then, since we are assuming that the centralizer of $\mathfrak{h}_{\mathfrak{a}}$ in $\mathfrak{g}^{\bar{0}}$ is $\mathfrak{h}_0$, we have that $(\tilde{b}_i)_{s_i-h}\in\mathfrak{h}_0$ and $\bar{b}^i_{-s_i+h}\in\bar{\mathfrak{h}}_0$. Thus, in this case $[f,(\tilde{b}_i)_{s_i-h}\bar{b}^i_{-s_i+h}]=0$.

For the second equality we can argue by induction on $|T|$, the result being obvious if $|T|=0$. To deal with the case $|T|>0$, recall that for $x\in\mathfrak{p}$, we write $\gamma(x)=\frac{1}{2}\sum_i:\overline{[x,b_i]}_{\mathfrak{p}}\overline{b}^i:$. Then, by (7.3), we have

$$[(G_{\mathfrak{g},\mathfrak{a}})_0,\overline{h}^T]=(\tilde{h}_0^{t_1}+\gamma(h^{t_1})_0)(\overline{h}_0^{t_2}\cdots\overline{h}_0^{t_k})+(\overline{h}_0^{t_1})[(G_{\mathfrak{g},\mathfrak{a}})_0,\overline{h}_0^{t_2}\cdots\overline{h}_0^{t_k}],$$

so, by the induction hypothesis, we need only to check that $\gamma(h)_{(0)}\equiv\rho_\sigma(h)$

mod $\overline{\mathcal{A}^1}$. We have

$$\begin{aligned}\gamma(h)_0 &\equiv -\frac{1}{2}\sum_i (\overline{b}^i)_0(\overline{[h,b_i]_{\mathfrak{p}}})_0 - \frac{1}{2}\sum_i (s_i+\frac{1}{2})([h,b_i],b^i) \qquad (8.34)\\ &\equiv \frac{1}{2}(-\sum_{b_i\in\mathfrak{n}_-\cap\mathfrak{p}}([h,b_i],b^i) - \sum_i (s_i+\frac{1}{2})([h,b_i],b^i))\\ &\equiv \frac{1}{2}(-\sum_{x_i\in\mathfrak{n}_-}([h,x_i],x^i) + \sum_{i,0<s_i\le\frac{1}{2}}(1-2s_i)([h,x_i],x^i))\\ &\equiv (h,\frac{1}{2}(-\sum_{x_i\in\mathfrak{n}_-}[x_i,x^i] + \sum_{i,0<s_i\le\frac{1}{2}}(1-2s_i)[x_i,x^i]))\\ &\equiv \rho_\sigma(h).\end{aligned}$$

□

**Lemma 8.8.** *The differential $\overline{d}_p$ maps $(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n$ to $(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n+1}$ and the induced differential on $Gr(\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})$ is $\partial_{p,0}$ (under the identification (8.29)).*

*Proof.* Fix $p\in\mathbb{N}$. It suffices to prove that, if $z+\overline{\mathcal{A}^{p+1}}=\mathcal{S}(z')$ with $z'\in K^p_n\otimes\mathcal{F}_{|C_{diag}}\otimes\wedge\mathfrak{h}_0$, then $[(G_{\mathfrak{g},\mathfrak{a}})_0,z]+\overline{\mathcal{A}^{p+1}}=\mathcal{S}(\partial_p(z'))+w$ with $w\in(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_n$.

Note that, if $w\in\mathcal{W}^k_{\mathcal{F}}$, we can find $u_1\in\mathcal{W}^k_{\mathcal{F}}$ and $u_2\in\overline{\mathcal{W}^k_{\mathcal{F}}F^{p+1}}$ such that $uw=u_1w+u_2$. This is easily checked using (8.8).

For $n\in\mathbb{N}$, let $\mathcal{W}^k_{\mathcal{F}}(p,n)$ be the span of monomials $\tilde{y}^I\bar{y}^J\tilde{h}^S f\bar{h}^T\tilde{x}^H\bar{x}^K$, with $\deg(H+K)\ge j_p$ and $|I|+|H|\le n$. Set also $\mathcal{W}^k_{\mathcal{F}}(n)=\mathcal{W}^k_{\mathcal{F}}(0,n)$. The ordinary PBW theorem for $\mathcal{W}$ combined with (8.5) shows that, if $u_1\in\mathcal{W}^k_{\mathcal{F}}(m)$ and $\deg(H+K)\ge j_p$, then

$$u_1(\tilde{y}^I\bar{y}^J\tilde{h}^S f\bar{h}^T\tilde{x}^H\bar{x}^K)\in\mathcal{W}^k_{\mathcal{F}}(p,m+|I|+|H|).$$

Moreover, if $\deg(u_1)=j_{q_1}$, $\deg(H_1+H_2+K_1+K_2)=j_{q_2}$, then

$$\tilde{y}^I\bar{y}^J\tilde{h}^S f\bar{h}^T\tilde{x}^{H_1}\bar{x}^{K_1}u_1\tilde{x}^{H_2}\bar{x}^{K_2}\in\mathcal{W}^k_{\mathcal{F}}(q_1+q_2,m+|H_1|+|H_2|+|I|).$$

We now apply the above observations to the computation of

$$[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{y}^I\overline{y}^J f\overline{h}^T\tilde{x}^H\overline{x}^K]$$

with $\deg(H+K)=j_p$ and $|H|+|I|=n$. Clearly

$$\begin{aligned}&[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{y}^I\overline{y}^J f\overline{h}^T\tilde{x}^H\overline{x}^K]=[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{y}^I]\overline{y}^J f\overline{h}^T\tilde{x}^H\overline{x}^K+\\ &\tilde{y}^I[(G_{\mathfrak{g},\mathfrak{a}})_0,\overline{y}^J]f\overline{h}^T\tilde{x}^H\overline{x}^K+(-1)^{|J|}\tilde{y}^I\overline{y}^J[(G_{\mathfrak{g},\mathfrak{a}})_0,f\overline{h}^T]\tilde{x}^H\overline{x}^K+\\ &+(-1)^{|J|+|T|}\tilde{y}^I\overline{y}^J f\overline{h}^T[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{x}^H]\overline{x}^K+(-1)^{|J|+|T|}\tilde{y}^I\overline{y}^J f\overline{h}^T\tilde{x}^H[(G_{\mathfrak{g},\mathfrak{a}})_0,\overline{x}^K].\end{aligned}$$

By the above observations, applying formula (7.2), we find that

$$[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{y}^I]\overline{y}^J f\overline{h}^T\tilde{x}^H\overline{x}^K \text{ and } \tilde{y}^I\overline{y}^J f\overline{h}^T[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{x}^H]\overline{x}^K \in \mathcal{W}^k_{\mathcal{F}}(p,n)+\overline{\mathcal{W}^k_{\mathcal{F}}F^{p+1}},$$

while Lemma 8.7 implies that

$$\tilde{y}^I\overline{y}^J[(G_{\mathfrak{g},\mathfrak{a}})_0, f\overline{h}^T]\tilde{x}^H\overline{x}^K \in \mathcal{W}^k_{\mathcal{F}}(p,n)+\overline{\mathcal{W}^k_{\mathcal{F}}F^{p+1}}.$$

It follows that

$$\begin{aligned}&[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{y}^I\overline{y}^J\overline{y}^S f\overline{h}^T\tilde{x}^H\overline{x}^K]+\overline{\mathcal{A}^{p+1}} \equiv \tilde{y}^I[(G_{\mathfrak{g},\mathfrak{a}})_0,\overline{y}^J]f\overline{h}^T\tilde{x}^H\overline{x}^K+\\ &(-1)^{|J|+|T|}\tilde{y}^I\overline{y}^J f\overline{h}^T\tilde{x}^H[(G_{\mathfrak{g},\mathfrak{a}})_0,\overline{x}^K]+\overline{\mathcal{A}^{p+1}} \mod (\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_n.\end{aligned}$$

Writing explicitly $\overline{x}^K=\overline{x}^{n_1}\cdots\overline{x}^{n_t}$ and $\overline{y}^J=\overline{y}^{m_1}\cdots\overline{y}^{m_t}$ we set $\overline{x}^{K_i}=\overline{x}^{n_1}\cdots\widehat{\overline{x}^{n_i}}\cdots\overline{x}^{n_t}$ and $\overline{y}^{J_i}=\overline{y}^{m_1}\cdots\widehat{\overline{y}^{n_i}}\cdots\overline{y}^{m_t}$. The above observations imply, by (7.3) and the fact that $[(G_{\mathfrak{g},\mathfrak{a}})_0,\bar{a}]=0$ if $a\in\mathfrak{a}$, that

$$\begin{aligned}&[(G_{\mathfrak{g},\mathfrak{a}})_0,\tilde{y}^I\overline{y}^J fh^T\tilde{x}^H\overline{x}^K]+\overline{\mathcal{A}^{p+1}} \equiv \sum_{i:y^{m_i}\in(\mathfrak{n}'_-)_{\mathfrak{p}}}(-1)^i\tilde{y}^I\tilde{y}^{m_i}\overline{y}^{J_i}f\overline{h}^T\tilde{x}^H\overline{x}^K\\ &\quad+\sum_{i:x^{m_i}\in(\mathfrak{n}')_{\mathfrak{p}}}(-1)^{|J|+|T|+i}\tilde{y}^I\overline{y}^J f\overline{h}^T\tilde{x}^{n_i}\tilde{x}^H\overline{x}^{K_i}+\overline{\mathcal{A}^{p+1}} \mod (\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_n\end{aligned}$$

as wished. □

We also need the following (possibly known) fact. It can be proved either generalizing Lemmas 4.3, 4.4 in [1] to the series case, or extending (as we do in Section 9.5) the standard homotopy argument which proves the exactness of the Koszul complex. Let $\mathcal{E}(z_1,\dots,z_n)$ denote the algebra of entire functions in $n$ complex variables $z_1,\dots,z_n$. Let $\xi$ be an odd variable and set $\xi_i=z_i\xi$.

**Lemma 8.9.** *Fix $h_0\in\mathbb{C}^n$. Consider the complex*

$$R^p=\mathcal{E}(z_1,\dots,z_n)\otimes\wedge^p(\xi_1,\dots,\xi_n)$$

*endowed with the Koszul differential*

$$\partial_{h_0}(f\otimes\xi_1\wedge\cdots\wedge\xi_p)=\sum_{k=1}^p(-1)^k(z_k-z_k(h_0))f\otimes\xi_1\wedge\cdots\wedge\widehat{\xi_k}\wedge\cdots\wedge\xi_p. \quad (8.35)$$

*Then $H_j(R^\bullet)=0$ for $j>0$. Moreover, given $f\in\mathcal{E}(z_1,\dots,z_n)$, then $f=f(h_0)+\partial_{h_0}(g)$ with $g\in R^1$.*

If $x^j = x_r$ with $x \in \mathfrak{a}^{\overline{r}}$, then we set $\tilde{x}^j_{\mathfrak{a}} = (\tilde{x})_{\mathfrak{a}r}$ and $\theta(x^j) = \theta(x)_r$ ($\theta$ is as in (3.18)). For a multi-index $J = (j_1, j_2, \dots)$ we set $\tilde{x}^J_{\mathfrak{a}} = x^{j_1}_{\mathfrak{a}} x^{j_2}_{\mathfrak{a}} \dots$. We define $y^j_{\mathfrak{a}}$ and $y^J_{\mathfrak{a}}$ similarly. We shall need the following technical lemma, whose proof is in Section 9.6.

**Lemma 8.10.**

*If* $\deg(H+K) = j_p$ *and* $|J+H| = n$ *then*

$$\overline{y}^I \tilde{y}^J_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K = \overline{y}^I \tilde{y}^J f \overline{h}^T \tilde{x}^H \overline{x}^K + u$$

*with* $u \in \overline{\mathcal{A}^p}$ *and* $u + \overline{\mathcal{A}^{p+1}} \in \left(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}\right)_{n-1}$.

The main use we will make of Lemma 8.10 is highlighted in the following Corollary.

**Corollary 8.11.** *If* $a = p \otimes f \otimes \wedge^T h$ *with* $p \in (K^{p,0}_n)_{\mathfrak{a}}$, $f \in (\mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_{\mathfrak{p}})$, *and* $\wedge^T h \in \wedge \mathfrak{h}_{\mathfrak{a}}$ *then*

$$\bar{d}_p(\mathcal{S}(a)) = \mathcal{S}(p \otimes \partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(f) \otimes \wedge^T h) + \bar{d}_p(v) + v'$$

*with* $v, v' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. *(See* (8.35) *for the definition of* $\partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}$*).*

*Proof.* Write explicitly $\mathcal{S}(a)$ as a sum of terms of type $\overline{y}^I \tilde{y}^J f \overline{h}^T \overline{h}^S \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}}$ with $f \in \mathcal{F}_{|C_{diag}}$, $\wedge^T h \in \wedge \mathfrak{h}_{\mathfrak{p}}$, and $\wedge^S h \in \wedge \mathfrak{h}_{\mathfrak{a}}$. Then, by Lemma 8.10, we have that $\overline{y}^I \tilde{y}^J f \overline{h}^T \overline{h}^S \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}} = \overline{y}^I \tilde{y}^J_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K + \overline{\mathcal{A}^{p+1}} + v$, with $v \in (\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{n-1}$. If $x \in \mathfrak{a}$ then $[d_{\mathfrak{a}}, \overline{(\tilde{x})_{\mathfrak{a}r}}] = r(\tilde{x})_{\mathfrak{a}r}$, hence we have that $[d_{\mathfrak{a}}, \overline{y}^I \tilde{y}^J_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K] = 0$, thus $v \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. Hence $\overline{d}_p(\mathcal{S}(a))$ is a sum of terms of type $\overline{y}^I \tilde{y}^J_{\mathfrak{a}}[(G_{\mathfrak{g},\mathfrak{a}})_0, f \overline{h}^T] \overline{h}^S \tilde{x}^H_{\mathfrak{a}} \overline{x}^K + \overline{\mathcal{A}^{p+1}} + \overline{d}_p(v)$. By Lemma 8.8 we obtain that $\overline{d}_p(\mathcal{S}(a))$ is a sum of terms of type $\overline{y}^I \tilde{y}^J_{\mathfrak{a}} \partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(f \overline{h}^T) \overline{h}^S \tilde{x}^H_{\mathfrak{a}} \overline{x}^K + \overline{\mathcal{A}^{p+1}} + \overline{d}_p(v)$. Applying Lemma 8.10 again, we find that $\overline{d}_p(\mathcal{S}(a))$ is a sum of terms of type $\overline{y}^I \tilde{y}^J \partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(f \overline{h}^T) \overline{h}^S \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}} + \overline{d}_p(v) + v'$, with $v, v' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. $\square$

Let $\mathcal{F}_{\mathfrak{a}}$ be the set of holomorphic functions on $\mathfrak{h}^*_{\mathfrak{a}} \oplus \mathbb{C}\delta_{\mathfrak{a}}$ viewed as a subset of $\mathcal{F}$. Consider the subalgebra $\mathcal{A}(\mathfrak{a})$ of $\overline{\mathcal{A}}$ generated by $\{a_r, \overline{a}_r \mid a \in \mathfrak{a}\} \cup \mathcal{F}_{\mathfrak{a}}$. Set $\mathcal{A}^p(\mathfrak{a}) = \mathcal{A}^p \cap \mathcal{A}(\mathfrak{a})$ and $\overline{\mathcal{A}^p_{inv}(\mathfrak{a})} = \overline{\mathcal{A}^p(\mathfrak{a})} \cap \overline{\mathcal{A}}^{inv}$. Notice that $[(G_{\mathfrak{g},\mathfrak{a}})_0, \overline{\mathcal{A}(\mathfrak{a})}] = 0$.

**Proposition 8.12.** *The embedding* $\overline{\mathcal{A}^p_{inv}(\mathfrak{a})} \to \overline{\mathcal{A}^p_{inv}}$ *induces an isomorphism*

$$\overline{\mathcal{A}^p_{inv}(\mathfrak{a})}/\overline{\mathcal{A}^{p+1}_{inv}(\mathfrak{a})} \simeq H(\overline{d}_p). \tag{8.36}$$

*Proof.* We will show by induction on $n$ that, if $x \in Ker\,\overline{d}_p \cap (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n$, then there is $a \in \overline{\mathcal{A}^p_{inv}(\mathfrak{a})}/\overline{\mathcal{A}^{p+1}_{inv}(\mathfrak{a})}$ and $b \in \overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}}$ such that $x = a + \overline{d}_p(b)$.

We set $(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{-1} = \{0\}$ so the result is obvious for $n = -1$. If $n \geq 0$ we have that

$$x = \sum_i \mathcal{S}(u_i) + x',$$

with $u_i \in K^{p,0}_n \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$ and $x' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. By Lemma 8.8, $\overline{d}_p(x) = \sum_i \mathcal{S}(\partial_{p,0}(u_i)) + x''$ with $x'' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n$ so, since $\overline{d}_p(x) = 0$, $\sum_i \mathcal{S}(\partial_{p,0}(u_i)) \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n$. Since $\partial_{p,0}(u_i) \in K^{p,0}_{n+1} \otimes \mathcal{F}_{|C_{diag}} \otimes \mathfrak{h}_0$, we have that $\partial_{p,0}(u_i) = 0$. By (8.31), we have that $u_i = a_i + \partial_{p,0}(u'_i)$ with $a_i \in K^{p,0}_{\mathfrak{a}} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$ and $u'_i \in \oplus_{m<n} K^{p,0}_m \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_0$. Set $b' = \sum \mathcal{S}(u'_i)$. It follows that

$$\overline{d}_p(b') = \sum \mathcal{S}(\partial_{p,0}(u'_i)) + b'' = \sum \mathcal{S}(u_i) + b'' - \sum \mathcal{S}(a_i)$$

with $b'' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. Hence $x = \sum \mathcal{S}(a_i) + \overline{d}_p(b') + u$ with $u \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$.

It follows that, if we write explicitly $a_i = \sum_T p_i \otimes f^i_T \otimes \wedge^T h$ with $p_i \in (K^{p,0}_n)_{\mathfrak{a}}$, $f^i_T \in \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_{\mathfrak{p}}$, $\wedge^T h \in \wedge\mathfrak{h}_{\mathfrak{a}}$, then, by Corollary 8.11,

$$\overline{d}_p(\sum \mathcal{S}(a_i)) = \sum \mathcal{S}(p_i \otimes \partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(f^i_T) \otimes \wedge^T h) + \overline{d}_p(v) + v'$$

with $v, v' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$.

Since $\overline{d}_p(x) = 0$ we see that $\overline{d}_p(u) = -\overline{d}_p(\sum \mathcal{S}(a_i))$, so

$$\sum \mathcal{S}(p_i \otimes \partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(f^i_T) \otimes \wedge^T h) = \overline{d}_p(-v - u) - v'.$$

Since $-v - u \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$, by Lemma 8.8, $\overline{d}_p(-v-u) = \sum \mathcal{S}(x_i) + u'$ with $x_i \in \sum_{\substack{r+s=p \\ m>0}} K^r_{\mathfrak{a}} \otimes (K^s_{\mathfrak{p}} \cap K^s_m) \otimes \mathcal{F}_{|C_{diag}} \otimes \mathfrak{h}_0$ and $u' \in (\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}})_{n-1}$, thus, comparing terms, we find

$$\partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(f^i_T) = 0.$$

Using Lemma 8.9 we can find $g^i_T \in \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_{\mathfrak{p}}$ such that $(f^i_T) = F^i_T + \partial_{-(\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{p}}}}(g^i_T)$. Here, if $f^i_T = \sum_L f^i_{T,L} \otimes \wedge^L h \in \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_{\mathfrak{p}}$, $F^i_T$ is the holomorphic function on $C_{diag}$ defined, for $\lambda \in \mathfrak{h}^*_{\mathfrak{p}}$ and $\mu = \overline{\mu} + \ell\delta_{\mathfrak{a}} \in \mathfrak{h}^*_{\mathfrak{a}} \oplus \mathbb{C}\delta_{\mathfrak{a}}$, by

$$F^i_T(\lambda + \overline{\mu} + \ell\delta + \rho_{\sigma\mathfrak{a}} - (\rho_\sigma)_{|\mathfrak{h}_{\mathfrak{a}}}, \mu) = f^i_{T,0}(\overline{\mu} + \ell\delta + \rho_{\sigma\mathfrak{a}} - \rho_\sigma, \mu).$$

Hence, by Corollary 8.11,

$$\sum \mathcal{S}(a_i) = \sum \mathcal{S}(p_i \otimes F_T^i \otimes \wedge^T h) + \overline{d}_p(\sum \mathcal{S}(p_i \otimes g_T^i \otimes \wedge^T h) + w) + w'$$

with $w, w' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. Let $\overline{F}^i_T \in \mathcal{F}_{\mathfrak{a}}$ be the function defined by $\overline{F}^i_T(\mu) = f^i_{T,0}(\overline{\mu} + \ell\delta + \rho_{\sigma\mathfrak{a}} - \rho_\sigma, \mu)$. Since $(\overline{F}^i_T)_{|C_{diag}} = F^i_T$ we obtain that $\sum \mathcal{S}(a_i) = \sum \mathcal{S}(p_i \otimes \overline{F}^i_T \otimes \wedge^T h) + \overline{d}_p(\sum \mathcal{S}(p_i \otimes g^i_T \otimes \wedge^T h) + w) + w'$. Writing explicitly $\mathcal{S}(p_i \otimes \overline{F}^i_T \otimes \wedge^T h)$ as a sum of terms of type $\overline{y}^I \tilde{y}^J \overline{F}^i_T \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}}$ we can apply Lemma 8.10 and write $\mathcal{S}(p_i \otimes \overline{F}^i_T \otimes \wedge^T h)$ as a sum of terms $\overline{y}^I \tilde{y}^J_{\mathfrak{a}} \overline{F}^i_T \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K + \overline{\mathcal{A}^{p+1}} + y$ with $y \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$. The final outcome is that $\sum \mathcal{S}(a_i) = a' + \overline{d}_p(z) + z'$ with $a' \in \overline{\mathcal{A}^p_{inv}(\mathfrak{a})}/\overline{\mathcal{A}^{p+1}_{inv}(\mathfrak{a})}$, $z' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$, hence

$$x = a' + \overline{d}_p(b' + z) + u + z'.$$

Since $\overline{d}_p(x) = 0$ we see that $\overline{d}_p(u + z') = 0$, so, by the induction hypothesis, there is $a'' \in \overline{\mathcal{A}^p_{inv}(\mathfrak{a})}/\overline{\mathcal{A}^{p+1}_{inv}(\mathfrak{a})}$ and $b''' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})$ such that $u + z' = a'' + \overline{d}_p(b''')$, therefore, setting $a = a' + a''$ and $b = b' + z + b'''$ we have that $x = a + \overline{d}_p(b)$.

We now prove by induction on $n$, that, if $a \in (\overline{\mathcal{A}^p_{inv}(\mathfrak{a})}/\overline{\mathcal{A}^{p+1}_{inv}(\mathfrak{a})}) \cap Im\, \overline{d}_p$ and $a \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_n$ then $a = 0$. First of all observe that $\overline{\mathcal{A}(\mathfrak{a})}$ is the closure of the algebra generated by $(\tilde{x})_{\mathfrak{a}r}, \overline{x}_r, f$ with $x \in \mathfrak{a}^{\overline{r}}$ and $f \in \mathcal{F}_{\mathfrak{a}}$. By applying the PBW theorem to $L'(\mathfrak{a},\sigma) \otimes Cl(L(\overline{\mathfrak{a}}, \overline{\tau})$ we can write $a$ as a sum of terms of type $\overline{y}^J \tilde{y}^J_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K + \overline{\mathcal{A}^{p+1}}$. By Lemma 8.10 we can assume that $\deg(K+H) = p$ and $|J+H| \le n$. Suppose now that $a = \overline{d}_p(u)$. We will show that $u \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-2}$ thus, by the induction hypothesis, $a = 0$. So assume that $u \in \sum \mathcal{S}(x_i) + u'$ with $x_i \in K^{p,0}_m \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_0$ and $u' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{m-1}$. Then, $a = \sum \mathcal{S}(\partial_{p,0}(x_i)) + v + \overline{d}_p(u')$ with $v \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_m$.

Since, by Lemma 8.8, $\partial_{p,0}(x_i) \in \sum_{\substack{r+s=p\\ m>0}} K^r_{\mathfrak{a}} \otimes (K^s_{\mathfrak{p}} \cap K^p_m) \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_0$, and, by Lemma 8.10, $a = \sum \mathcal{S}(a_i) + a'$ with $a_i \in (K^{p,0}_n)_{\mathfrak{a}} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_0$ and $a' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{n-1}$, by comparing terms, we deduce that, if $m \ge n-1$, then $\partial_{p,0}(x_i) = 0$.

Since $\partial_{p,0}$ is exact except in degree 0, we see that $x_i = a'_i + \partial_{p,0}(y_i)$ with $a'_i \in K^{p,0}_{\mathfrak{a}} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_0$ and $y_i \in K^{p,0}_{m-1} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_0$. Setting $y = \sum \mathcal{S}(y_i)$ we have that $\overline{d}_p(y) = \sum \mathcal{S}(x_i) - \sum \mathcal{S}(a'_i) + y'$ with $y' \in (\overline{\mathcal{A}^p_{inv}}/\overline{\mathcal{A}^{p+1}_{inv}})_{m-1}$, hence $u = \overline{d}_p(y) - y' + \sum \mathcal{S}(a'_i) + u'$. Substituting $u$ with $u - \overline{d}_p(y)$ and $u'$ with $u' - y'$, we can assume that

$$u = \sum \mathcal{S}(a'_i) + u', \tag{8.37}$$

with $a'_i \in (K_m^{p,0})_{\mathfrak{a}} \otimes \mathcal{F}_{|C_{diag}} \otimes \wedge \mathfrak{h}_0$.

Setting $a'_i = \sum_T p_i \otimes f_T^i \otimes \wedge^T h^T$ and applying Corollary 8.11, we obtain that

$$\overline{d}_p(u) = \sum \mathcal{S}(p_i \otimes \partial_{-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}}(f_T^i) \otimes \wedge^T h) + \overline{d}_p(v') + v''$$

with $v', v'' \in (\overline{\mathcal{A}_{inv}^p}/\overline{\mathcal{A}_{inv}^{p+1}})_{m-1}$. Since $\bar{d}_p(u) = a$, comparing terms we deduce that $\partial_{-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}}(f_T^i) \in (\mathcal{F}_{\mathfrak{a}})_{|C_{diag}}$, this means that it is a function on $C_{diag}$ that does not depend from $\lambda \in \mathfrak{h}_\mathfrak{p}$. On the other hand, since $\partial_{-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}}(f_T^i)$ is a boundary, it is zero when computed in $-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}$ hence $\partial_{-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}}(f_T^i) = 0$. Arguing as in the first part of the proof, we obtain that $\sum \mathcal{S}(a'_i) = a' + \overline{d}_p(z) + z'$ with $a' \in (\overline{\mathcal{A}_{inv}^p(\mathfrak{a})}/\overline{\mathcal{A}_{inv}^{p+1}(\mathfrak{a})})$ and $z' \in (\overline{\mathcal{A}_{inv}^p}/\overline{\mathcal{A}_{inv}^{p+1}})_{m-1}$. Since $\overline{d}_p(a') = 0$ we find that $\overline{d}_p(u - \overline{d}_p(z) - a') = a$, hence, in light of (8.37), we can substitute $u$ with $z' + u'$ and assume $u \in (\overline{\mathcal{A}_{inv}^p}/\overline{\mathcal{A}_{inv}^{p+1}})_{m-1}$. Repeating the argument we can assume $m < n - 1$ and conclude the proof. □

We now come to the main reduction.

**Corollary 8.13.** *The embedding $\overline{\mathcal{A}_{inv}(\mathfrak{a})} \to \overline{\mathcal{A}}^{inv}$ induces an isomorphism*

$$\overline{\mathcal{A}_{inv}(\mathfrak{a})} \simeq H(\mathbf{d}).$$

*Proof.* Fix $x \in \overline{\mathcal{A}}^{inv}$ such that $\mathbf{d}(x) = 0$. Then $\overline{d}_0(x + \overline{\mathcal{A}_{inv}^1}) = 0$, so, by (8.36), $x + \overline{\mathcal{A}_{inv}^1} = a^0 + \overline{\mathcal{A}_{inv}^1} + \overline{d}_0(v^0 + \overline{\mathcal{A}_{inv}^1})$. That is $x = a^0 + \mathbf{d}(v^0) + u^1$ with $a^0 \in \overline{\mathcal{A}_{inv}(\mathfrak{a})}$, $v^0 \in \overline{\mathcal{A}}^{inv}$, and $u^1 \in \overline{\mathcal{A}_{inv}^1}$. Since $\mathbf{d}(a^0) = 0$ we see that $\mathbf{d}(u^1) = 0$, hence, arguing as above, we can write $u^1 = a^1 + \mathbf{d}(v^1) + u^2$, with $a^1 \in \overline{\mathcal{A}_{inv}^1(\mathfrak{a})}$, $v^1 \in \overline{\mathcal{A}_{inv}^1}$, and $u^2 \in \overline{\mathcal{A}_{inv}^2}$. Substituting we find that $x = a^0 + a^1 + \mathbf{d}(v^0 + v^1) + u^2$. An obvious induction shows then that for any $n$ we can find $a^n, v^n, u^{n+1}$ such that $a^n \in \overline{\mathcal{A}_{inv}^n(\mathfrak{a})}$, $v^n \in \overline{\mathcal{A}_{inv}^n}$, and $u^{n+1} \in \overline{\mathcal{A}_{inv}^{n+1}}$ and

$$x = \sum_{i=0}^n a^i + \mathbf{d}(\sum_{i=0}^n v_i) + u^{n+1}.$$

Since $\lim_i a^i = \lim_i v^i = \lim_i u^i = 0$, setting $a = \sum_{i=0}^\infty a^i$ and $v = \sum_{i=0}^\infty v^i$, we have

$$x = a + \mathbf{d}(v).$$

Suppose now that $a \in \overline{\mathcal{A}_{inv}(\mathfrak{a})}$ is such that $a \in Im\, \mathbf{d}$. Then $a + \overline{\mathcal{A}_{inv}^1} \in Im\, \overline{d}_0$, hence, by (8.36), $a \in \overline{\mathcal{A}_{inv}^1}$. Repeating this argument we find that $a \in \overline{\mathcal{A}_{inv}^p}$ for any $p$, thus $a = 0$. □

## 8.5 The final step

We need to recall the main results of [7]. Let $\mathcal{L} \subset \widehat{\mathfrak{h}}_0^*$ be the degeneracy locus of the Shapovalov form. Let $\mathfrak{F}$ be the set of complex-valued functions on $\widehat{\mathfrak{h}}_0^* \setminus \mathcal{L}$. In [7] it is proven that there exists a natural algebra structure on $U_{\mathfrak{F}} = U(\widehat{L}(\mathfrak{g}, \sigma)) \otimes_{U(\widehat{\mathfrak{h}}_0)} \mathfrak{F}$ and on a completion $\widehat{U}_{\mathfrak{F}}$. Roughly speaking, $\widehat{U}_{\mathfrak{F}}$ consists of suitable series $\sum_{I,J} y^I \varphi_{I,J} x^J$, with $y^I, x^J$ monomials in negative and positive root vectors with respect to a triangular decomposition of $\widehat{L}(\mathfrak{g}, \sigma)$ and $\varphi_{I,J} \in \mathfrak{F}$. Recall that $C_{\mathfrak{g}}$ denotes the Tits cone of $\widehat{L}(\mathfrak{g}, \sigma)$ (see (8.1)).

**Proposition 8.14.** *Let $Z_{\mathfrak{F}}$ denote the center of $\widehat{U}_{\mathfrak{F}}$.*

1. *Given $\varphi \in \mathfrak{F}$ there exists a unique element $z_\varphi = \sum_{I,J} y^I \varphi_{I,J} x^J \in Z_{\mathfrak{F}}$ such that $\varphi_{0,0} = \varphi$.*

2. *The "Harish-Chandra" map $H : Z_{\mathfrak{F}} \to \mathfrak{F}, H(z_\varphi) = \varphi$ is an algebra isomorphism.*

3. *Let $\varphi \in \mathfrak{F}$ a function which can be extended to an holomorphic function on $-\widehat{\rho}_\sigma + C_{\mathfrak{g}}$. Let $\varphi_{-\widehat{\rho}_\sigma}$ be the holomorphic function on $C_{\mathfrak{g}}$ defined by $\varphi_{-\widehat{\rho}_\sigma}(\lambda) = \varphi(\lambda - \widehat{\rho}_\sigma)$. If $\varphi_{-\widehat{\rho}_\sigma}$ is $\widehat{W}$-invariant, then $z_\varphi$ can be extended to an holomorphic element (i.e. $z_\varphi = \sum_{I,J} y^I \varphi_{I,J} x^J$ with all the $\varphi_{I,J}$ holomorphic on $-\widehat{\rho}_\sigma + C_{\mathfrak{g}}$).*

Let $\mathfrak{F}_{hol}$ be the subalgebra of $\mathfrak{F}$ of the functions that can be extended to holomorphic functions on $-\widehat{\rho}_\sigma + C_{\mathfrak{g}}$. Set $U_{\mathfrak{F}_{hol}} = U(\widehat{L}(\mathfrak{g}, \sigma)) \otimes_{S(\widehat{\mathfrak{h}}_0)} \mathfrak{F}_{hol} \subset U_{\mathfrak{F}}$. Note that the embedding

$$U(L'(\mathfrak{g}, \sigma)) \otimes \mathfrak{F}_{hol} \subset U(\widehat{L}(\mathfrak{g}, \sigma)) \otimes \mathfrak{F}_{hol}$$

induces an algebra isomorphism

$$U(L'(\mathfrak{g}, \sigma)) \otimes_{S(\mathfrak{h}'_0)} \mathfrak{F}_{hol} \simeq U_{\mathfrak{F}_{hol}}. \tag{8.38}$$

As in Section 8.3, if we choose a basis $\{x^i\}$ of $\mathfrak{n}'$ and a basis $\{y^i\}$ of $\mathfrak{n}'_-$, and $I$ is a multi-index, we use the notation $x^I$, $y^I$ to denote the corresponding monomials. Assume that $x^i$ and $y^i$ are root vectors of $\widehat{L}(\mathfrak{g}, \sigma)$ and let $\gamma_i$ be the root corresponding to $x^i$. Clearly we can assume that $-\gamma_i$ is the root corresponding to $y^i$. If $I = (i_1, i_2, \dots)$, let $ht(I) = \sum_j i_j ht(\gamma_j)$. Consider the subalgebra $\widehat{U}^0_{\mathfrak{F}_{hol}}$ of $\widehat{U}_{\mathfrak{F}}$ whose elements are series $\sum y^I \phi_{I,J} x^J$ with $\phi_{I,J} \in \mathfrak{F}_{hol}$ and $\sum i_h \gamma_h = \sum j_h \gamma_h$.

If $\phi \in \mathfrak{F}_{hol}$, let $f_\phi \in \mathcal{F}$ be defined by

$$f_\phi(\lambda, \mu) = \phi(k\Lambda_0 + \lambda). \tag{8.39}$$

Then the mapping $x \otimes \phi \mapsto \tilde{x} f_\phi$ defines an algebra map from $U(L'(\mathfrak{g}, \sigma)) \otimes \mathfrak{F}_{hol}$ to $\overline{\mathcal{A}}$. Note that the map pushes down to define, thanks to the isomorphism (8.38), a map

$$\Omega : U_{\mathfrak{F}_{hol}} \to \overline{\mathcal{A}}. \tag{8.40}$$

We claim that $\Omega$ can be extended to a map

$$\Omega : \widehat{U}^0_{\mathfrak{F}_{hol}} \to \overline{\mathcal{A}}.$$

Indeed, recall from the beginning of § 8.2 that $\alpha_i(\hat{f}_\mathfrak{a}) > 0$ for all simple roots $\alpha_i$. Set $C = \inf(\alpha_i(\hat{f}_\mathfrak{a}))$. Then, if $\lambda = \sum_i n_i \alpha_i$, $n_i \in \mathbb{N}$, we have $\sum_i n_i \alpha_i(\hat{f}_\mathfrak{a}) \geq C \sum_i n_i$. Thus $\deg(\tilde{x}^J) \geq C\, ht(J)$, hence the series $\sum \tilde{y}^I \phi_{I,J} \tilde{x}^J$ converges in $\overline{\mathcal{A}}$.

Recall that given a level $k$ highest weight module $M$ for $\widehat{L}(\mathfrak{g}, \sigma)$, then an action of $\widehat{U}_\mathfrak{F}$ on $M$ is defined in [7]. Moreover $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ decomposes as a direct sum of weight spaces both for $\widehat{\mathfrak{h}}_0$ and $\widehat{\mathfrak{h}}_\mathfrak{a}$. Since the actions of $\widehat{\mathfrak{h}}_0$ and $\widehat{\mathfrak{h}}_\mathfrak{a}$ commute, we can write

$$M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}}) = \bigoplus_{(\lambda,\mu) \in \widehat{\mathfrak{h}}_0^* \times \widehat{\mathfrak{h}}_\mathfrak{a}^*} (M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}}))_{(\lambda,\mu)}.$$

Then, if $(M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}}))_{(\lambda,\mu)} \neq \{0\}$, we have that $\lambda = k\Lambda_0 + \overline{\lambda} + x\delta$ and $\mu = \sum_S (k + g - g_S)\Lambda_0^S + \overline{\mu} + y\delta_\mathfrak{a}$ with $\overline{\mu} \in \mathfrak{h}_\mathfrak{a}^*$. We can therefore define an action of $\mathcal{F}$ on $M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ by setting $f \cdot v = f(\overline{\lambda} + x\delta, \overline{\mu} + y\delta_\mathfrak{a})v$ for $v \in (M \otimes F^{\overline{\tau}}(\overline{\mathfrak{g}}))_{(\lambda,\mu)}$. We can then extend this action to $\overline{\mathcal{A}}$. Observe that for any $v \in M$, $w \in F^{\overline{\tau}}(\overline{\mathfrak{p}})$ and $z \in \widehat{U}^0_{\mathfrak{F}_{hol}}$

$$\Omega(z)(v \otimes w) = (zv) \otimes w. \tag{8.41}$$

We finally come to the main result of this section.

*Proof of Theorem 8.1.* Let $f$ be as in the statement and let $\phi$ be the holomorphic function on $-\widehat{\rho}_\sigma + C_\mathfrak{g}$ defined by $\phi(\lambda) = f(\lambda + \widehat{\rho}_\sigma)$. Extend $\phi$ to an element of $\mathfrak{F}_{hol}$. By part 3 of Proposition 8.14 we get the existence of a central element $z_\phi$ of $\widehat{U}_\mathfrak{F}$ such that $z_\phi \in \widehat{U}^0_{\mathfrak{F}_{hol}}$ and $z_\phi \cdot v = f(\Lambda + \widehat{\rho}_\sigma)v$ for any $v \in M$. Moreover, $z_\phi = \phi + \sum_{I \neq 0, J \neq 0} y^I \phi_{I,J} x^J$. It follows that $\Omega(z_\phi) = f_\phi + x$ (cf. (8.39)) with $x \in \overline{\mathcal{A}^1_{inv}}$. Recall that we are viewing $\mathcal{F}_\mathfrak{a}$ as a subset of $\mathcal{F}$. Let $\Phi \in \mathcal{F}_\mathfrak{a}$ be the function defined in $\mu = \overline{\mu} + x\delta_\mathfrak{a}$, by

$$\Phi(\mu) = \phi(k\Lambda_0 + \overline{\mu} + x\delta + \rho_{\sigma\mathfrak{a}} - \rho_\sigma).$$

By Lemma 8.9, there is $f' \in \mathcal{F}_{|C_{diag}} \otimes \wedge\mathfrak{h}_\mathfrak{p}$ such that $(f_\phi)_{|C_{diag}} = f_0 + \partial_{-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}}(f')$ where $f_0$ is the function on $C_{diag}$ defined, for $\lambda \in \mathfrak{h}_\mathfrak{p}^*$, $\mu \in \mathfrak{h}_\mathfrak{a}^*$, by

$$f_0(\lambda+\mu+\rho_{\sigma\mathfrak{a}}-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{a}}+x\delta, \mu+x\delta_\mathfrak{a}) = f_\phi(\mu+\rho_{\sigma\mathfrak{a}}-\rho_\sigma+x\delta, \mu+x\delta_\mathfrak{a}) = \Phi(\mu+x\delta_\mathfrak{a}).$$

Hence $f_0 = \Phi_{|C_{diag}}$.

According to Lemma 8.7, in $\overline{\mathcal{A}^0}/\overline{\mathcal{A}^1}$, $\mathcal{S}(\partial_{-(\rho_\sigma)_{|\mathfrak{h}_\mathfrak{p}}}(f')) = \bar{d}_0(\mathcal{S}(f'))$ hence we can write $f_\phi + \overline{\mathcal{A}^1_{inv}} = f_0 + \overline{\mathcal{A}^1} + \bar{d}_0(\mathcal{S}(f'))$. Thus $f_\phi + \overline{\mathcal{A}^1_{inv}} = \Phi + \overline{\mathcal{A}^1_{inv}} + \bar{d}_0(w)$ with $w \in \overline{\mathcal{A}^0_{inv}}/\overline{\mathcal{A}^1_{inv}}$. Since $\Omega(z_\phi) + \overline{\mathcal{A}^1_{inv}} = f_\phi + \overline{\mathcal{A}^1_{inv}}$, we have that $\Omega(z_\phi) = \Phi + \mathbf{d}(z) + z_1$ with $z_1 \in \overline{\mathcal{A}^1_{inv}}$. Since $z_\phi$ is central, $[(G_{\mathfrak{g},\mathfrak{a}})_0, \Omega(z_\phi)] = 0$, hence $\mathbf{d}(\Omega(z_\phi) - \Phi - \mathbf{d}(z)) = 0$. It follows that $\mathbf{d}(z_1) = 0$. According to Corollary 8.13, there is $a \in \overline{\mathcal{A}^1_{inv}(\mathfrak{a})}$ such that $z_1 = a + \mathbf{d}(y)$, hence, setting $u = y + z$

$$\Omega(z_\phi) = \Phi + a + \mathbf{d}(u).$$

Suppose now that a highest weight $\widehat{L}(\mathfrak{a},\sigma)$-module $Y$ of highest weight $\mu = \sum_S (k+g-g_S)\Lambda_0^S + \overline{\mu} + y\delta_\mathfrak{a}$ occurs in the Dirac cohomology $H((G_{\mathfrak{g},\mathfrak{a}})_0^{N'})$ of $N' = M \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ and that $v$ is an highest vector for $Y$. Since $a \in \overline{\mathcal{A}^1_{inv}(\mathfrak{a})}$, by Lemma 8.10, $a \cdot v = 0$, hence, by (8.41)

$$\Omega(z_\phi)v = f(\Lambda + \widehat{\rho}_\sigma)v = \Phi(\mu)v.$$

hence $f(\Lambda + \widehat{\rho}_\sigma) = \Phi(\mu) = \phi(k\Lambda_0 + \overline{\mu} + y\delta + \rho_{\sigma\mathfrak{a}} - \rho_\sigma)$. Finally observe that $\phi(k\Lambda_0+\overline{\mu}+y\delta+\rho_{\sigma\mathfrak{a}}-\rho_\sigma) = f((k+g)\Lambda_0+\overline{\mu}+y\delta+\rho_{\sigma\mathfrak{a}}) = f((\varphi_\mathfrak{a}^*)^{-1}(\mu+\widehat{\rho}_{\mathfrak{a}\sigma})) = f_{|\widehat{\mathfrak{h}}_\mathfrak{a}^*}(\mu + \widehat{\rho}_{\mathfrak{a}\sigma})$. $\square$

# 9 Proofs of technical results

## 9.1 Evaluation of $[G_{\mathfrak{g}\lambda}G_\mathfrak{g}]$

Let $G_\mathfrak{g} \in V^{k+g,1}(R^{super})$ be the affine Dirac operator defined in (4.1); write for shortness $G$ instead of $G_\mathfrak{g}$. We want to calculate $[G_\lambda G]$ by using expression (4.5). We assume that $\mathfrak{g}$ is reductive. We let $\mathfrak{g} = \sum_S \mathfrak{g}_S$ be the eigenspace decomposition of $\mathfrak{g}$ with respect to the Casimir operator $Cas$ of $\mathfrak{g}$ and let $2g_S$ be the eigenvalue relative to $\mathfrak{g}_S$.

We proceed in steps. First of all we fix an orthonormal basis $\{x_i^S\}$ of $\mathfrak{g}_S$ and choose $\{x_i\} = \cup_S \{x_i^S\}$ as orthonormal basis of $\mathfrak{g}$. If $a \in \mathfrak{g}_S$,

$$[\tilde{a}_\lambda G] = \sum_i :\widetilde{[a,x_i]}\overline{x}_i: + \lambda :(k+g-g_S)\overline{a}:, \tag{9.1}$$

and

$$[G_\lambda \tilde{a}] = -\sum_i :\widetilde{[a,x_i]}\overline{x}_i: + \lambda :(k+g-g_S)\overline{a}: + :(k+g-g_S)T(\overline{a}):, \tag{9.2}$$

$$\begin{aligned}[G_\lambda : \tilde{a}\overline{a} :] = -\sum_i :\widetilde{[a,x_i]}\overline{x}_i\overline{a}: + :\tilde{a}a: \\ + :(k+g-g_S)T(\overline{a})\overline{a}: + \frac{\lambda^2}{2}(k+g-g_S)(a,a).\end{aligned} \tag{9.3}$$

Formula (9.1) is checked directly whereas (9.2) follows from (9.1) by the sesquilinearity of the $\lambda$-bracket. Finally, (9.3) follows from (9.2) using the Wick formula (2.5).

$$[\overline{a}_\lambda : \tilde{b}\overline{b} :] = (a,b)\tilde{b} \tag{9.4}$$

$$[: \tilde{b}\overline{b} :_\lambda \overline{a}] = (a,b)\tilde{b} \tag{9.5}$$

$$[: \tilde{a}\overline{a} :_\lambda : \overline{b}\overline{c} :] = (a,b) : \tilde{a}\overline{c} : -(a,c) : \tilde{a}\overline{b} : \tag{9.6}$$

$$[: \tilde{a}\overline{a} :_\lambda : \overline{b}\overline{c}\overline{d} :] = (a,b) : \tilde{a}\overline{c}\overline{d} : -(a,c) : \overline{b}\tilde{a}\overline{d} : +(a,d) : \overline{b}\overline{c}\tilde{a} : \tag{9.7}$$

$$= (a,b) : \tilde{a}\overline{c}\overline{d} : -(a,c) : \tilde{a}\overline{b}\overline{d} : +(a,d) : \tilde{a}\overline{b}\overline{c} : \tag{9.8}$$

In checking all the formulas one uses the Wick formula (2.5). Moreover, (9.4) follows from(2.9), (9.5) follows from (9.4) by sesquilinearity. Formula (9.6) is checked similarly. Finally one proves (9.7) combining (9.5) and(9.6). The equality between (9.7) and (9.8) follows from 2.3 and relations $: \tilde{a}\overline{b} :=: \overline{b}\tilde{a} :$ and $: \overline{a}\tilde{b}\overline{c} :=: \tilde{b}\overline{a}\overline{c} :$. From (9.8) it follows also that

$$\sum_{h,k}[: \tilde{a}\overline{a} :_\lambda : \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :] = -3\sum_h : \tilde{a}\overline{[a,x_h]}\overline{x}_h : \tag{9.9}$$

In turn (9.3) and (9.9) imply

$$\begin{aligned}\sum_i [G_\lambda : \tilde{x}_i\overline{x}_i :] = \sum_{i,j} : \widetilde{[x_i,x_j]}\overline{x}_i\overline{x}_j : + \sum_{i,S} : (k+g-g_S)T(\overline{x}_i^S)\overline{x}_i^S : \\ + \sum_i : \tilde{x}_i\overline{x}_i : + \sum_S \frac{\lambda^2}{2}(k+g-g_S)\dim\mathfrak{g}_S.\end{aligned} \tag{9.10}$$

$$\sum_{i,h,k}[: \tilde{x}_i\overline{x}_i :_\lambda : \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :] = -3\sum_{i,k} : \tilde{x}_i\overline{[x_i,x_k]}\overline{x}_k : . \tag{9.11}$$

We start now computing $\sum_{i,j,h,k}[:\overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda: \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :]$. We have

$$\sum_{i,j}[\overline{a}_\lambda : \overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :] = -3\sum_i :\overline{[a,x_i]}\overline{x}_i : . \tag{9.12}$$

$$\sum_{i,j}[:\overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda \overline{a}] = -3\sum_i :\overline{[a,x_i]}\overline{x}_i : . \tag{9.13}$$

$$\sum_i[\overline{a}_\lambda : \overline{[b,x_i]}\overline{x}_i :] = 2\overline{[a,b]}. \tag{9.14}$$

$$\sum_i[:\overline{[a,x_i]}\overline{x}_i :_\lambda \overline{b}] = 2\overline{[a,b]}. \tag{9.15}$$

Hence

$$\begin{aligned}
&\sum_{i,j,h,k}[:\overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda: \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :] \\
&= -3\sum_{i,h,k}[::\overline{[[x_h,x_k],x_i]}\overline{x}_i : \overline{x}_h\overline{x}_k : - \sum_{i,j,h,k} :\overline{[x_h,x_k]}[:\overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda: \overline{x}_h\overline{x}_k :] \\
&- 3\sum_{i,h,k}\int_0^\lambda [:\overline{[[x_h,x_k],x_i]}\overline{x}_i :_\mu: \overline{x}_h\overline{x}_k :]d\mu \\
&= -3\sum_{i,h,k} ::\overline{[[x_h,x_k],x_i]}\overline{x}_i : \overline{x}_h\overline{x}_k : +3\sum_{i,h,k} :\overline{[x_h,x_k]} : \overline{[x_h,x_i]}\overline{x}_i : \overline{x}_k : \\
&- 3\sum_{i,h,k} :\overline{[x_h,x_k]}\overline{x}_h\overline{[x_k,x_i]}\overline{x}_i : \\
&- \sum_{i,j,h,k}\int_0^\lambda :\overline{[x_h,x_k]}[[:\overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda \overline{x}_h]_\mu\overline{x}_k] : d\mu \qquad (9.16) \\
&- 6\sum_{h,k}\int_0^\lambda [:\overline{[[x_h,x_k],x_h]}\overline{x}_k : d\mu - 6\sum_{h,k}\int_0^\lambda [:\overline{x}_h\overline{[[x_h,x_k],x_k]} : d\mu \\
&- 3\sum_{i,h,k}\int_0^\lambda\int_0^\mu [[:\overline{[[x_h,x_k],x_i]}\overline{x}_i :_\mu \overline{x}_h]_\nu\overline{x}_k :]d\nu d\mu \qquad (9.17)
\end{aligned}$$

Using the relation $\sum_h[x_h,x_k]\cdot[x_h,x_j] = \sum_h[[x_j,x_h],x_k]\cdot x_h$ for any bilinear

product $\cdot$, we can rewrite (9.17) as

$$
\begin{aligned}
&-3\sum_{i,h,k} :: \overline{[[x_h,x_k],x_i]}\overline{x}_i : \overline{x}_h\overline{x}_k : +3\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]} : \overline{x}_h\overline{x}_i : \overline{x}_k : \\
&-3\sum_{i,h,k} : \overline{[x_h,[x_i,x_k]]}\overline{x}_h\overline{x}_k\overline{x}_i : +6\sum_{i,h,k}\int_0^\lambda : \overline{[x_h,x_k][x_h,x_k]} : d\mu \\
&-6\sum_{h,k}\int_0^\lambda : \overline{[x_k,x_h][x_k,x_h]} : d\mu - 6\sum_{h,k}\int_0^\lambda : \overline{[x_k,x_h][x_h,x_k]} : d\mu \\
&+12\frac{\lambda^2}{2}\sum_S g_S \dim \mathfrak{g}_S
\end{aligned}
$$

Since $:\overline{aa}:= 0$ (by (2.2)) we get

$$
\begin{aligned}
&-3\sum_{i,h,k} :: \overline{[[x_h,x_k],x_i]}\overline{x}_i : \overline{x}_h\overline{x}_k : +3\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]} : \overline{x}_h\overline{x}_i : \overline{x}_k : \\
&-3\sum_{i,h,k} : \overline{[x_h,[x_i,x_k]]}\overline{x}_h\overline{x}_k\overline{x}_i : +12\frac{\lambda^2}{2}\sum_S g_S dim \mathfrak{g}_S.
\end{aligned}
$$

By (2.2) and (2.3) we find

$$
\sum_{i,h,k} :: \overline{[[x_h,x_k],x_i]}\overline{x}_i : \overline{x}_h\overline{x}_k := \sum_{i,h,k} : \overline{[[x_h,x_k],x_i]}\overline{x}_i\overline{x}_h\overline{x}_k : -8\sum_{S,i} g_S : T(\overline{x}_i^S)\overline{x}_i^S :
$$

hence

$$
\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]} : \overline{x}_h\overline{x}_i : \overline{x}_k := -\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]} : \overline{x}_i\overline{x}_h : \overline{x}_k :,
$$

and therefore

$$
\begin{aligned}
&\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]} : \overline{x}_h\overline{x}_i : \overline{x}_k := \\
&-\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]}\overline{x}_i\overline{x}_h\overline{x}_k : +4\sum_{S,i} g_S : T(\overline{x}_i^S)\overline{x}_i^S :
\end{aligned}
$$

Upon substituting we find

$$
\begin{aligned}
&-3\sum_{i,h,k} : \overline{[[x_h,x_k],x_i]}\overline{x}_i\overline{x}_h\overline{x}_k : -3\sum_{i,h,k} : \overline{[[x_i,x_h],x_k]}\overline{x}_i\overline{x}_h\overline{x}_k : \\
&-3\sum_{i,h,k} : \overline{[x_h,[x_i,x_k]]}\overline{x}_h\overline{x}_k\overline{x}_i : +36\sum_{i,S} g_S : T(\overline{x}_i^S)\overline{x}_i^S : +12\frac{\lambda^2}{2}\sum_S g_S dim \mathfrak{g}_S
\end{aligned}
$$

By the Jacobi identity we find

$$\sum_{i,j,h,k}[:\overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda : \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :] = 36\sum_{S,i} g_S : T(\overline{x}_i^S)\overline{x}_i^S : +\lambda^2\sum_S \frac{2g_S}{3} dim\mathfrak{g}_S.$$

Hence

$$\begin{aligned}[G_\lambda G] &= \sum_i [G_\lambda : \tilde{x}_i\overline{x}_i :] - \frac{1}{6}\sum_{i,j,h}[G_\lambda : \overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :] \\ &= \sum_i [G_\lambda : \tilde{x}_i\overline{x}_i :] - \frac{1}{6}\sum_{i,h,k}[: \tilde{x}_i\overline{x}_i :_\lambda : \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :] \\ &+ \frac{1}{36}\sum_{i,j,h,k}[: \overline{[x_i,x_j]}\overline{x}_i\overline{x}_j :_\lambda : \overline{[x_h,x_k]}\overline{x}_h\overline{x}_k :],\end{aligned}$$

which in turn gives

$$\begin{aligned}[G_\lambda G] &= \sum_{i,j} : \widetilde{[x_i,x_j]}\overline{x}_i\overline{x}_j : + \sum_{S,i} : (k+g-g_S)T(\overline{x}_i^S)\overline{x}_i^S : \\ &+ \sum_i : \tilde{x}_i x_i : + \frac{\lambda^2}{2}\sum_S (k+g-g_S)\dim\mathfrak{g}_S \\ &+ \frac{1}{2}\sum_{i,k} : \tilde{x}_i\overline{[x_i,x_k]}\overline{x}_k : \\ &+ \sum_{S,i} g_S : T(\overline{x}_i^S)\overline{x}_i^S : + \frac{\lambda^2}{2}\sum_S \frac{g_S}{3}\dim\mathfrak{g}_S\end{aligned}$$

or

$$\begin{aligned}[G_\lambda G] &= \sum_{i,j} : \widetilde{[x_i,x_j]}\overline{x}_i\overline{x}_j : + (k+g)\sum_i : T(\overline{x}_i)\overline{x}_i : \\ &+ \sum_i : \tilde{x}_i\tilde{x}_i : + \sum_{i,k} : \tilde{x}_i\overline{[x_i,x_k]}\overline{x}_k : \\ &+ \frac{\lambda^2}{2}\sum_S (k+g-\frac{2g_S}{3})\dim\mathfrak{g}_S\end{aligned}$$

Since

$$\sum_{i,j} : \widetilde{[x_i,x_j]}\overline{x}_i\overline{x}_j := -\sum_{i,j} : \tilde{x}_i\overline{[x_i,x_j]}\overline{x}_j :$$

we finally get

**Proposition 9.1.**

$$[G_\lambda G] = \sum_i :\tilde{x}_i\tilde{x}_i: +(k+g)\sum_i :T(\overline{x}_i)\overline{x}_i: +\frac{\lambda^2}{2}\sum_S (k+g-\frac{2g_S}{3})\dim \mathfrak{g}_S. \tag{9.18}$$

## 9.2 $(G_{\mathfrak{g},\mathfrak{a}})_0$ is selfadjoint

Assume that $\mathfrak{g}$ is semisimple. If $\{\alpha_i\}$ is the set of simple roots of $\widehat{L}(\mathfrak{g},\sigma)$, let $\mathfrak{h}_\mathbb{R}$ be the real span of $\{\frac{h_{\overline{\alpha}_i}}{(\overline{\alpha}_i,\overline{\alpha}_i)}\} \subset \mathfrak{h}_0$. Set $\mathfrak{t}_0 = i\mathfrak{h}_\mathbb{R}$ and let $\mathfrak{k}$ be a compact form of $\mathfrak{g}$. By Exercise 8 of Chapter VI of [4], we can assume that $\mathfrak{k}$ is $\sigma$-invariant and that $\mathfrak{t}_0 \subset \mathfrak{k}$. Denote by *conj* the conjugation in $\mathfrak{g}$ w.r.t. $\mathfrak{k}$: $conj(x+iy) = x-iy$, $x,y\in\mathfrak{k}$, and let $\omega_0$ be the antilinear antiautomorphism of $\mathfrak{g}$ defined by $\omega_0(x) = -conj(x)$. Extend $\omega_0$ to an antiautomorphism of $\widehat{L}(\mathfrak{g},\sigma)$ by

$$\omega_0(x_{(r)}) = (\omega_0(x))_{(-r)}, \quad \omega_0(K) = K, \quad \omega_0(d) = d.$$

If $\alpha_i$ is a simple root of $\widehat{L}(\mathfrak{g},\sigma)$, with $\alpha_i = s_i\delta+\overline{\alpha}_i$, choose $x_i \in \mathfrak{g}^{s_i}\cap\mathfrak{g}_{\overline{\alpha}_i}$ in such a way that $[x_i,\omega_0(x_i)] = \frac{2h_{\overline{\alpha}_i}}{(\overline{\alpha}_i,\overline{\alpha}_i)}$. Set $e_i = t^{s_i}\otimes x_i$, $f_i = \omega_0(e_i)$, so that $[e_i,f_i] = \alpha_i^\vee = \frac{2h_{\overline{\alpha}_i}}{(\overline{\alpha}_i,\overline{\alpha}_i)} + \frac{2s_i}{(\overline{\alpha}_i,\overline{\alpha}_i)}K$. With this choice of Chevalley generators for $\widehat{L}(\mathfrak{g},\sigma)$, $\omega_0$ is the unique antilinear antiautomorphism of $\widehat{L}(\mathfrak{g},\sigma)$ which exchanges the $e_i$ with the $f_i$ and which is the identity when restricted to $\widehat{\mathfrak{h}}_\mathbb{R} = span_\mathbb{R}(\alpha_i^\vee)$. Recall that a $\widehat{L}(\mathfrak{g},\sigma)$-module $M$ is called unitarizable if there is a positive definite hermitian form $H(\cdot,\cdot)$ on $M$ such that $H(x\cdot v,w) = H(v,\omega_0(x)\cdot w)$. Theorem 11.7 of [8] asserts that $L(\Lambda)$ is unitarizable if and only if $\Lambda$ is dominant integral.

Recall that we are assuming that $g\in\mathbb{R}^+$ hence $(\cdot,\cdot)$ is negative definite on $\mathfrak{k}$. Thus we can define a positive definite hermitian form $H(\cdot,\cdot)$ on $\mathfrak{g}$ by the formula

$$H(x,y) = (x,\omega_0(y)). \tag{9.19}$$

Define a positive definite hermitian form $\overline{H}(\cdot,\cdot)$ on $L(\overline{\mathfrak{g}},\tau)$ by

$$\overline{H}(x_{(r)},y_{(s)}) = \delta_{r,s}H(x,y).$$

We can extend this form on the Clifford module $F^{\overline{\tau}}(\overline{\mathfrak{g}})$ by setting

$$H(x_1\cdots x_n\cdot 1, y_1\cdots y_m\cdot 1) = \delta_{n,m}\det(H(x_i,y_j))$$

where it is again positive definite. A standard calculation shows that, if $\overline{x}\in\overline{\mathfrak{g}}$ and $v,w\in F^{\overline{\tau}}(\overline{\mathfrak{g}})$

$$\overline{H}(x_{(r-\frac{1}{2})}v,w) = \overline{H}(v,(\omega_0(x))_{(-r-\frac{1}{2})}w).$$

Suppose that $M$ is a unitarizable $\widehat{L}(\mathfrak{g},\sigma)$-module. On $N = M\otimes F^{\overline{\tau}}(\mathfrak{g})$ we can consider the form $\{\cdot,\cdot\} = H(\cdot,\cdot)\otimes \overline{H}(\cdot,\cdot)$. Straightforward calculations show that for $v,w\in M\otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$

$$\begin{aligned}\{\sum_i (:\tilde{x}_i\overline{x}_i:)^{\overline{\tau}}_{(\frac{1}{2})}v,w\} &= \{v,\sum_i (:\widetilde{\omega_0(x_i)}\overline{\omega_0(x_i)}:)^{\overline{\tau}}_{(\frac{1}{2})}\cdot w\}\\ &= \{v,\sum_i (:\tilde{x}_i\overline{x}_i:)^{\overline{\tau}}_{(\frac{1}{2})}\cdot w\},\end{aligned}$$

and similarly for the cubic term of $(G_{\mathfrak{g}})^N_0$. In particular, if $\Lambda$ is dominant integral, then $(G_{\mathfrak{g}})^N_0$ is self-adjoint with respect to a Hermitian positive definite form on $L(\Lambda)\otimes F^{\overline{\tau}}(\overline{\mathfrak{g}})$ . For the general case of $G_{\mathfrak{g},\mathfrak{a}}$ it suffices to apply the argument to $(G_{\mathfrak{a}})^N_0$ acting on $N = (L(\Lambda)\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}))\otimes F^{\overline{\tau}}(\overline{\mathfrak{a}})$ and to note that $L(\Lambda)\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ is unitarizable since, as shown in § 5.2, $F^{\overline{\tau}}(\overline{\mathfrak{p}})$ is the restriction to $\widehat{L}(\mathfrak{a},\sigma)$ of a unitarizable representation of $\widehat{L}(so(\mathfrak{p}),Ad(\sigma))$. Summing up

**Proposition 9.2.** *The action of $(G_{\mathfrak{g},\mathfrak{a}})^{N'}_0$ on $N' = L(\Lambda)\otimes F^{\overline{\tau}}(\overline{\mathfrak{p}})$ is self-adjoint with respect to the positive definite Hermitian form $\{\ ,\ \}$. Consequently, $Ker(G_{\mathfrak{g},\mathfrak{a}})^{N'}_0 = Ker((G_{\mathfrak{g},\mathfrak{a}})^{N'}_0)^2$.*

## 9.3 Proof of Lemma 8.2

If $\Lambda\in -\widehat{\rho}_\sigma + C_{\mathfrak{g}}$ is a weight of level $k$, let $M(\Lambda)$ be the Verma module of highest weight $\Lambda$. Let $(\cdot,\cdot)_\Lambda$ be the Shapovalov form on $M(\Lambda)$ and $\omega$ the Chevalley involution of $\widehat{L}(\mathfrak{g},\sigma)$. Choose $\widehat{\rho}\in\widehat{\mathfrak{h}}_0^*$ such that $\widehat{\rho}(\alpha_i^\vee)=1$ for all $i$. We assume furthermore that $2(\Lambda+\widehat{\rho}+\gamma,\alpha)\neq n(\alpha,\alpha)$ for all $n>0$, $\gamma$ in the root lattice of $\widehat{L}(\mathfrak{g},\sigma)$, and $\alpha\in\widehat{\Delta}^+$. With this assumption we have that $(\cdot,\cdot)_\Lambda$ is nondegenerate (see e.g. [7, Lemma 1]). It follows that, for each $\mu=\sum n_i\alpha_i$ with $n_i\in\mathbb{N}$ there is a basis $\{\mathbf{y}^{\mathbf{i}}_\mu\}$ of $U(\mathfrak{n}'_-)_{-\mu}$ such that $\{\mathbf{y}^{\mathbf{i}}_\mu v_\Lambda\}$ is an orthonormal basis of $M(\Lambda)_{\Lambda-\mu}$. Here and in the following we use boldface letters to make clear that we are dealing with polynomials and multi-indexes. Set $\mathbf{x}^{\mathbf{i}}_\mu=\omega(\mathbf{y}^{\mathbf{i}}_\mu)$. Note that $\mathbf{x}^{\mathbf{i}}_\mu\mathbf{y}^{\mathbf{j}}_\nu\cdot v_\Lambda\in\bigoplus_{ht(\eta)=ht(\nu)-ht(\mu)} M(\Lambda)_{\Lambda-\eta}$. In particular, if $ht(\mu)=ht(\nu)$, we have that $\mathbf{x}^{\mathbf{i}}_\mu\mathbf{y}^{\mathbf{j}}_\nu\cdot v_\Lambda\in\mathbb{C}v_\Lambda$. Since $(\mathbf{x}^{\mathbf{i}}_\mu\mathbf{y}^{\mathbf{j}}_\nu\cdot v_\Lambda,v_\Lambda)_\Lambda = (\mathbf{y}^{\mathbf{j}}_\nu\cdot v_\Lambda,\mathbf{y}^{\mathbf{i}}_\mu\cdot v_\Lambda)_\Lambda=\delta_{\nu,\mu}\delta_{i,j}$, we see that

$$\mathbf{x}^{\mathbf{i}}_\mu\mathbf{y}^{\mathbf{j}}_\nu\cdot v_\Lambda=\delta_{\nu,\mu}\delta_{i,j}v_\Lambda. \tag{9.20}$$

Fix a basis $\{h_i\}$ of $\mathfrak{h}_0$ such that $(h_i,h_{n-j+1})=\delta_{ij}$ and set

$$\mathfrak{h}_0^+ = span(h_i\mid i\leq\lfloor\frac{n}{2}\rfloor),\quad \mathfrak{h}_0^- = span(h_i\mid i\geq\lceil\frac{n}{2}\rceil+1).$$

Note that $\mathfrak{h}_0 = \mathfrak{h}_0^+ \oplus \mathfrak{h}_0^- \oplus E$ with $dim E \le 1$. Set $\overline{\mathfrak{g}}^+ = \overline{\mathfrak{h}}_0^+ \oplus \overline{\mathfrak{n}}'$ and $\overline{\mathfrak{g}}^- = \overline{\mathfrak{h}}_0^- \oplus \overline{\mathfrak{n}}'_-$. If $s \in \mathbb{N}$, set $Cl(\overline{\mathfrak{g}}^\pm)_s = (\overline{\mathfrak{g}}^\pm)^s \subset Cl(\overline{\mathfrak{g}})$. Note that $Cl(\overline{\mathfrak{g}}^+)_s Cl(\overline{\mathfrak{g}}^-)_r \cdot 1 \in \oplus_{t \le r-s} Cl(\overline{\mathfrak{g}}^-)_t \cdot 1$. Moreover it is easy to construct a basis $\{\mathbf{u}^{\mathbf{i}s}\}$ of $Cl(\overline{\mathfrak{g}}^+)_s$ and a basis $\{\mathbf{v}^{\mathbf{j}s}\}$ of $Cl(\overline{\mathfrak{g}}^-)_s$ such that

$$\mathbf{u}^{\mathbf{i}s}\mathbf{v}^{\mathbf{j}s} \cdot 1 = \delta_{ij}. \tag{9.21}$$

Suppose that $u \in \overline{\mathcal{W}^k \otimes 1}$ is such that

$$u \cdot v = 0 \quad \text{for any } v \in M(\Lambda) \otimes F^{\overline{\tau}}(\overline{\mathfrak{p}}).$$

Write $u = \lim \pi(u_n)$. Fix $p > 0$. Since $\pi(u_n)$ is a Cauchy sequence we can find $M$ such that, for any $n \ge M$, $\pi(u_M - u_n) \in \mathcal{W}^k F^p$. Suppose that $\deg(\mathbf{x}^{\mathbf{j}}{}_\nu \otimes \mathbf{u}^{\mathbf{s}t}) < p$. Choose $n \ge M$ big enough so that $u_n \cdot (\mathbf{y}^{\mathbf{j}}_\nu \cdot v_\Lambda \otimes \mathbf{v}^{\mathbf{s}t} \cdot 1) = 0$. Then $u_M \cdot (\mathbf{y}^{\mathbf{j}}_\nu \cdot v_\Lambda \otimes \mathbf{v}^{\mathbf{s}t} \cdot 1) = (u_n - (u_n - u_M)) \cdot (\mathbf{y}^{\mathbf{j}}_\nu \cdot v_\Lambda \otimes \mathbf{v}^{\mathbf{s}t} \cdot 1) = 0$. Hence, for any $\mathbf{j}, \mathbf{s}, t, \nu$ such that $\deg(\mathbf{x}^{\mathbf{j}}{}_\nu \otimes \mathbf{u}^{\mathbf{s}t}) < p$,

$$u_M \cdot (\mathbf{y}^{\mathbf{j}}_\nu \cdot v_\Lambda \otimes \mathbf{v}^{\mathbf{s}t} \cdot 1) = 0. \tag{9.22}$$

Using the PBW theorem we can write

$$u_M = \sum_{j,\nu,t,s,\epsilon} c_{\mathbf{j},\nu,\mathbf{t},s,\epsilon} w^\epsilon (\mathbf{x}^{\mathbf{j}}_\nu \otimes \mathbf{u}^{\mathbf{t}s}),$$

with $c_{\mathbf{j},\nu,\mathbf{t},s,\epsilon} \in U(\mathfrak{n}'_- \oplus \mathfrak{h}'_0) \otimes Cl(\bar{\mathfrak{g}}^-)$. Here $w^\epsilon$ occurs only if $\dim E = 1$ and, in such a case, $E = \mathbb{C}w$ and $\epsilon \in \{0, 1\}$.

We now show by induction on $q$ that $c_{\mathbf{j},\nu,\mathbf{t},s,\epsilon} w^\epsilon \in \mathcal{W}(K - k)$ for all $\mathbf{j}, \nu, \mathbf{t}, s, \epsilon$ such that $ht(\nu) + s = q$ and $\deg(\mathbf{x}^{\mathbf{j}}_\nu \otimes \mathbf{u}^{\mathbf{t}s}) < p$. This concludes the proof since it implies that $\pi(u_M) \in \mathcal{W}^k F^p$, hence $\pi(u_n) \in \mathcal{W}^k F^p$ for $n \ge M$.

If $ht(\nu) = s = 0$, then $\pi(u_M) \cdot v_\Lambda \otimes 1 = u_M \cdot v_\Lambda \otimes 1 = 0$. Hence $\sum_\epsilon c_{1,0,1,0,\epsilon} w^\epsilon \cdot v_\Lambda \otimes 1 = 0$. Write explicitly $c_{1,0,1,0,\epsilon} = \sum_{\mathbf{i},\mu,\mathbf{t},n} (\mathbf{y}^{\mathbf{i}}_\mu \otimes \mathbf{v}^{\mathbf{t}n}) p_{\mathbf{i},\mu,\mathbf{t},n}$, with $p_{\mathbf{i},\mu,\mathbf{t},n} \in U(\mathfrak{h}'_0)$. Since $U(\mathfrak{n}'_-) \otimes Cl(\overline{\mathfrak{g}}^- \oplus N)$ acts freely on $v_\Lambda \otimes 1$ (and $U(\mathfrak{h}'_0)$ commutes with $w^\epsilon$), we see that $p_{\mathbf{i},\mu,\mathbf{t},n}(k\Lambda_0 + \overline{\Lambda}) = 0$ for any $\mathbf{i}, \mu, \mathbf{t}, n$. Since $\overline{\Lambda}$ can be chosen in a dense subset of $\mathfrak{h}_0$, we see that $p_{\mathbf{i},\mu,\mathbf{t},n}(k\Lambda_0 + \overline{\Lambda}) = 0$ for any $\overline{\Lambda}$, thus $K - k$ divides $p_{\mathbf{i},\mu,\mathbf{t},n}$.

Fix now $\nu_0$ and $s_0$ with $ht(\nu_0) + s_0 = q > 0$ and such that $\deg(\mathbf{x}^{\mathbf{l}}_{\nu_0} \otimes \mathbf{u}^{\mathbf{m}s_0}) < p$. Applying (9.22) we get that

$$\sum_{\substack{\mathbf{j},\nu,s,\mathbf{t},\epsilon \\ ht(\nu)+s \ge q}} c_{j,\nu,s,t,\epsilon} w^\epsilon (\mathbf{x}^{\mathbf{j}}_\nu \otimes \mathbf{u}^{\mathbf{t}s}) \cdot (\mathbf{y}^{\mathbf{l}}_{\nu_0} \cdot v_\Lambda \otimes \mathbf{v}^{\mathbf{m}s_0} \cdot 1) = 0.$$

Indeed, by the induction hypothesis, in the above sum appear only coefficients with $ht(\nu) + s \ge q$. Since $\mathbf{x}^{\mathbf{j}}_\nu \mathbf{y}^{\mathbf{l}}_{\nu_0} \cdot v_\Lambda = 0$ if $ht(\nu) > ht(\nu_0)$ and $\mathbf{u}^{\mathbf{t}s}\mathbf{v}^{\mathbf{m}s_0} \cdot 1 = 0$

if $s > s_0$, we can write

$$\sum_{\substack{\mathbf{j},\nu,s,\mathbf{t},\epsilon \\ ht(\nu)=ht(\nu_0),s=s_0}} c_{\mathbf{j},\nu,s,\mathbf{t},\epsilon} w^\epsilon(\mathbf{x}_\nu^{\mathbf{j}} \otimes \mathbf{u}^{\mathbf{t}s}) \cdot (\mathbf{y}_{\nu_0}^{\mathbf{l}} \cdot v_\Lambda \otimes \mathbf{v}^{\mathbf{m}s_0} \cdot 1) = 0.$$

Using (9.20), (9.21), we get

$$\sum_\epsilon c_{\mathbf{l},\nu_0,\mathbf{m},s_0,\epsilon} w^\epsilon \cdot (v_\Lambda \otimes 1) = 0.$$

Arguing as above we deduce that $c_{\mathbf{l},\nu_0,\mathbf{m},s_0,\epsilon} \in \mathcal{W}(K-k)$.

## 9.4 Proof of Lemma 8.3

We may assume that $a =: T^{i_1}(x^1)\cdots T^{i_h}(x^h) :$ with $x^i \in \{\tilde{x}, \overline{x} \mid x \in \mathfrak{g}_\alpha^{\overline{r}}, r \in \mathbb{R}, \alpha \in \mathfrak{h}_0^*\}$. Set $N(a,r) = cr + \deg(a)$ and $U(a,r) = \overline{\mathcal{W}^k F^{N(a,r)}}$. We will prove by induction on $h$ that for each $r$ there is $a_r \in U(a,r)$ such that $a_r \cdot v = a_r^N \cdot v$.

If $h = 1$, we set

$$T^i(\tilde{x})_r = (-1)^i i! \binom{r+i}{i} t^r \otimes x, \quad T^i(\overline{x})_r = (-1)^i i! \binom{r+i-\frac{1}{2}}{i} t^{r-\frac{1}{2}} \otimes \overline{x}. \tag{9.23}$$

If $h > 1$ we can assume that $a =: T^{i_1}(x^1) b :$ with $b =: T^{i_2}(x^2)\cdots T^{i_h}(x^h) :$. By Wick formula, we have that $T^{i_1}(x^1)_{(j)} b = \sum : T^{j_1}(y^1)\cdots T^{j_v}(y^v) :$ with $v < h$, so, by the induction hypothesis, we can define

$$(T^{i_1}(x^1)_{(j)} b)_r = \sum : T^{j_1}(y^1)\cdots T^{j_v}(y^v) :_r . \tag{9.24}$$

Choose $m \in \overline{r}_{x^1} - \Delta_{x^1}$ and set

$$\begin{aligned} a_r = &-\sum_{j\geq 1} \binom{m+\Delta_{x^1}+i_1-1}{j} (T^{i_1}(x^1)_{(j-1)} b)_r \\ &+ \sum_{j=0}^{\infty} T^{i_1}(x^1)_{m-j-1} b_{r-m+j+1} + p(x^1,b) b_{r-m-j} T^{i_1}(x^1)_{m+j}. \end{aligned}$$

By (9.23)

$$b_{r-m-j} T^{i_1}(x^1)_{m+j} \in U(x^1, m+j)$$

and, by the induction hypothesis,

$$T^{i_1}(x^1)_{m-j-1} b_{r-m+j+1} \in U(b, r-m+j+1),$$

hence the series

$$\sum_{j=0}^{\infty} T^{i_1}(x^1)_{m-j-1} b_{r-m+j+1} + p(x^1, b) b_{r-m-j} T^{i_1}(x^1)_{m+j}$$

converges. Therefore the definition of $a_r$ makes sense and, by (3.4), $a_r \cdot v = a_r^N \cdot v$ for any $N$, $v \in N$.

To conclude the induction step we need to check that $a_r \in U(a, r)$. Recall that we wrote $T^{i_1}(x^1)_{(j)} b = \sum_{v<h} : T^{j_1}(y^1) \cdots T^{j_v}(y^v) :$, hence we can assume that $\deg(: T^{j_1}(y^1) \cdots T^{j_v}(y^v) :) = \deg(a)$. By the induction hypothesis it follows that $(T^{i_1}(x^1)_{(j)} b)_r \in U(a, r)$.

It follows from (3.1) and Lemma 8.2, that

$$[T^{i_1}(x^1)_s, b_{r-s}] = \sum_{j \geq 0} \binom{s + \Delta_{x^1} + i_1 - 1}{j} (T^{i_1}(x^1)_{(j)} b)_r. \tag{9.25}$$

The induction hypothesis and (9.25) now imply that

$$T^{i_1}(x^1)_s b_{r-s} \in U(a, r), \quad b_{r-s} T^{i_1}(x^1)_s \in U(a, r)$$

for, if $\deg(b) + c(r - s) < \deg(a) + cr$, then $\deg(x^1) + cs > 0$ and

$$T^{i_1}(x^1)_s b_{r-s} = b_{r-s} T^{i_1}(x^1)_s + p(x^1, b)[T^{i_1}(x^1)_s, b_{r-s}]$$

and, if $\deg(x^1) + cs < \deg(a) + cr$, then

$$b_{r-s} T^{i_1}(x^1)_s = T^{i_1}(x^1)_s b_{r-s} + p(x^1, b)[b_{r-s}, T^{i_1}(x^1)_s].$$

This concludes the induction step.

Finally (8.9) follows easily by induction on $h$ and the explicit formula for $a_r$.

## 9.5 Proof of Lemma 8.9

We may clearly assume that $h_0 = 0$. Set for shortness $\partial = \partial_0$. Note that $\partial = \sum_{i=1}^n z_i \frac{\partial}{\partial \xi_i}$. Define $h = \sum_{i=1}^n \xi_i \frac{\partial}{\partial z_i}$, and note that both $\partial$ and $h$ are odd derivations. Hence we have that

$$h\partial + \partial h = \sum_{i,j=1}^n [z_i \frac{\partial}{\partial \xi_i}, \xi_j \frac{\partial}{\partial z_j}] = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i} + \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}.$$

Denote by $E = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$ the Euler operator and take a $p$-cycle $f \otimes u \in R_p$ with $p > 0$; we have $(h\partial + \partial h)(f \otimes u) = (E(f) + pf) \otimes u$. Define $\varphi = \sum \frac{f_{i_1,\dots,i_n}}{i_1+\cdots+i_n+p} z_1^{i_1} \cdots z_n^{i_n}$. Then $\varphi$ is holomorphic in any polydisk $|z_i| \leq r_i$, since

$$\sum \frac{|f_{i_1,\dots,i_n}|}{i_1 + \cdots + i_n + p} r_1^{i_1} \cdots r_n^{i_n} \leq \sum |f_{i_1,\dots,i_n}| r_1^{i_1} \cdots r_n^{i_n}.$$

Set $M = \sum_{i=1}^n z_i \mathcal{E}(z_1, \dots, z_n) + \sum_{p>0} \mathcal{E}(z_1, \dots, z_n) \otimes \wedge^p(\xi_1, \dots, \xi_n)$. Then $E + p\,I : M \to M$ is bijective, and $M$ is preserved by the Koszul operator. Moreover $\partial$ commutes with $\partial h + h\partial$. Therefore $\varphi \otimes u$ is a cycle if $f \otimes u$ is, hence

$$f = (E + p\,I)(\varphi \otimes u) = (h\partial + \partial h)(\varphi \otimes u) = \partial(h(\varphi \otimes u))$$

as required.

## 9.6 Proof of Lemma 8.10

We first show that if $\deg(I) = j_p$ and $|I| = n$ then

$$\tilde{x}^I_{\mathfrak{a}} = \tilde{x}^I + u \tag{9.26}$$

with $u \in \overline{\mathcal{A}^p}$ and $u + \overline{\mathcal{A}^{p+1}} \in \left(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}\right)_{n-1}$.

We will show by induction on $|I|$ that $\tilde{x}^I_{\mathfrak{a}} = \tilde{x}^I + u$ with $u$ a series $\sum_{i=p}^{\infty} u_i$ where $u_i = \sum c^i_{J,T,H,K} \overline{y}^J \overline{h}^T \tilde{x}^H \overline{x}^K$ and $c^i_{J,T,H,K} \in \mathbb{C}$, $\deg(H+K) = j_i$, $|H| < n$.

If $|I| = 0$ the result is obvious. Suppose $|I| = n > 0$. Then

$$\tilde{x}^I_{\mathfrak{a}} = \tilde{x}^{i_1}_{\mathfrak{a}} \tilde{x}^{I_0}_{\mathfrak{a}} = \tilde{x}^{i_1} \tilde{x}^{I_0}_{\mathfrak{a}} + \theta(x^{i_1}) \tilde{x}^{I_0}_{\mathfrak{a}},$$

with $|I_0| = n - 1$. If $\deg(I_0) = j_{p'}$, applying the induction hypothesis, we have

$$\tilde{x}^I_{\mathfrak{a}} = \tilde{x}^I + \tilde{x}^{i_1} u' + \theta(x^{i_1}) \tilde{x}^{I_0} + \theta(x^{i_1}) u',$$

where $u'$ is a series $\sum_{i=p'}^{\infty} u'_i$ with $u'_i = \sum c'^i_{J,T,H,K} \overline{y}^J \overline{h}^T \tilde{x}^H \overline{x}^K$, $deg(H + K) = j_i$, and $|H| < n - 1$.

Let $u = \tilde{x}^{h_1} u' + \theta(x^{i_1}) \tilde{x}^{I_0} + \theta(x^{i_1}) u'$. Clearly $\tilde{x}^{i_1} u'_i = \sum d^i_{J,T,H,K} \overline{y}^J \overline{h}^T \tilde{x}^H \overline{x}^K$ with $\deg(H+K) = j_i + \deg(x^{i_1}) \geq p' + \deg(x^{i_1}) = j_p$ and, by PBW theorem, $|H| < n$. Since $\deg(\theta(x^{i_1})) = \deg(\tilde{x}^{i_1})$, by the explicit expression (3.19) for $\theta(x^{i_1})$, we see that $\theta(x^{i_1}) = \sum_{i=i_0}^{\infty} \theta_i$ where $j_{i_0} = \deg(x^{i_1})$ and $\theta_i = \sum k^i_{J,T,K} \overline{y}^J \overline{h}^T \overline{x}^K$ with $k^i_{J,T,K} \in \mathbb{C}$, $\deg(K) = j_i$.

Hence $\theta(x^{i_1}) \tilde{x}^{I_0} = \sum_{i=i_0}^{\infty} \theta_i \tilde{x}^{I_0}$ and $\theta_i \tilde{x}^{I_0} = \sum k^i_{J,T,K} \overline{y}^J \overline{h}^T \tilde{x}^{I_0} \overline{x}^K$. Moreover $\deg(I_0 + K) = \deg(I_0 + i) \geq \deg(I_0) + \deg(x^{i_1}) = j_p$ and $|I_0| < n$.

Finally $\theta(x^{i_1})u' = \sum_{i=p'}^{\infty} \theta(x^{i_1})u'_i$. Note that $\theta(x^{i_1})u'_i = \sum_{t=i}^{\infty} u''_{it}$ with $u''_{it} = \sum c^{it}_{J,T,H,K}\overline{y}^I \overline{h}^T \tilde{x}^H \overline{x}^K$ with $\deg(H+K) = j_t$ and $|H| < n$. Since $\deg((\tilde{x})_{\mathfrak{a}r}) = \deg(\tilde{x}_r)$ we have that $\deg(\tilde{x}^{I_0}_{\mathfrak{a}}) = \deg(\tilde{x}^{I_0})$, hence $\deg(u') = \deg(I_0)$. It follows that $\deg(\theta(x^{i_1})u'_i) = j_p$. This implies, by Remark 8.2, that $\sum_{i=p'}^{t} u''_{it} = 0$ for $t < p$. This concludes the induction step and proves (9.26).

We prove the statement of the Lemma by induction on $|J|$. If $|J| = 0$ the statement is easily obtained from the previous step. If $|J| > 0$ then

$$\overline{y}^I \tilde{y}^J_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K = \overline{y}^I \tilde{y}^{j_1} \tilde{y}^{J_0}_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K + \overline{y}^I \theta(y^{j_1}) y^{J_0}_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K.$$

Applying the induction hypothesis we can write

$$\overline{y}^I \tilde{y}^J_{\mathfrak{a}} f \overline{h}^T \tilde{x}^H_{\mathfrak{a}} \overline{x}^K = \overline{y}^I \tilde{y}^J f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{y}^I \tilde{y}^{j_1} u' + \overline{y}^I \theta(y^{j_1}) \tilde{y}^{J_0} f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{y}^I \theta(y^{j_1}) u'.$$

with $u' \in \overline{\mathcal{A}^p}$ and $u' + \overline{\mathcal{A}^{p+1}} \in \left(\overline{\mathcal{A}^p}/\overline{\mathcal{A}^{p+1}}\right)_{n-2}$.

Set $u = \overline{y}^I \tilde{y}^{j_1} u' + \overline{y}^I \theta(y^{j_1}) \tilde{y}^{J_0} f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{y}^I \theta(y^{j_1}) u'$. Clearly $u \in \overline{\mathcal{A}^p}$. Write $u' = \sum \overline{y}^{I'} \tilde{y}^{J'} f_{I',J',T',H',K'} \overline{h}^{T'} \tilde{x}^{H'} \overline{x}^{K'} + u''$ with $\deg(H' + K') = j_p$, $|J' + H'| < n-1$, and $u'' \in \overline{\mathcal{A}^{p+1}}$.

By PBW theorem

$$\begin{aligned}\overline{y}^I \tilde{y}^{j_1} u' + \overline{\mathcal{A}^{p+1}} &= \sum \overline{y}^I y^{I'} \tilde{y}^{j_1} \tilde{y}^{J'} f_{I',J',T',H',K'} \overline{h}^{T'} \tilde{x}^{H'} \overline{x}^{K'} + \overline{\mathcal{A}^{p+1}} \\ &= \sum \overline{y}^{I''} \tilde{y}^{J''} f_{I'',J'',T',H',K'} \overline{h}^{T'} \tilde{x}^{H'} \overline{x}^{K'} + \overline{\mathcal{A}^{p+1}}\end{aligned}$$

with $|J'' + H'| < n$.

Writing, as above, $\theta(y^{j_1}) = \sum_{i=0}^{\infty} \theta_i$ with $\theta_i = \sum k^i_{J,T,K} \overline{y}^J \overline{h}^T \overline{x}^K$ with $k^i_{J,T,K} \in \mathbb{C}$, $\deg(K) = j_i$ and using the commutation relations (8.5) and (8.6), we see that

$$\overline{y}^I \theta(y^{j_1}) \tilde{y}^{J_0} f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}} = \overline{y}^I \theta_0 \tilde{y}^{J_0} f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}}$$

hence

$$\overline{y}^I \theta(y^{j_1}) \tilde{y}^{J_0} f \overline{h}^T \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}} = \sum \overline{y}^{I'} \tilde{y}^{J_0} f_{I',T'} \overline{h}^{T'} \tilde{x}^H \overline{x}^K + \overline{\mathcal{A}^{p+1}}$$

and, clearly, $|J_0 + H| < n$.

The same argument shows that

$$\overline{y}^I \theta(y^{j_1}) u' + \overline{\mathcal{A}^{p+1}} = \sum \overline{y}^{I'} \tilde{y}^{J'} f_{I',J',T',H',K'} \overline{h}^{T'} \tilde{x}^{H'} \overline{x}^{K'} + \overline{\mathcal{A}^{p+1}}$$

with $|J' + H'| < n$ as desired.

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